An analytic solution for the transition from a highly viscous fluid to a rigid solid and its relation to thermodynamic principles

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Outline

- Introduction
- Setting a model problem to calculate the entropy production
- Outlook
Outline

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- Setting a model problem to calculate the entropy production
- Outlook
Introduction - I

• Goal: Predicting a process in a thermodynamic proper way, hence find five primary field quantities in space-time for a continua

mass density \( \rho(x,t) \)
velocities \( v_i(x,t) \)
temperature \( T(x,t) \)

• \( \rightarrow \) initial boundary value problem based on conservation laws

balance of mass \( \frac{d\rho}{dt} + \rho \frac{\partial v_i}{\partial x_i} = 0 \) \( \frac{d\rho}{dt} + \rho v_i \frac{\partial v_i}{\partial x_i} = \rho f_i \)
balance of linear momentum \( \rho \frac{d v_i}{dt} + \frac{\partial \sigma_{ji}}{\partial x_j} = \rho f_i \) \( \sigma_{ij}, q_i, u, \rho, v_i \)
balance of internal energy \( \rho \frac{du}{dt} + \rho \frac{\partial q_i}{\partial x_i} = \sigma_{ij} \frac{\partial v_i}{\partial x_j} + \rho r \)

\( \sigma_{ij}, q_i, u, \rho, v_i \)

How to find relations for these?
Introduction - II

- Goal:
  Find out the constitutive relations for irreversible processes of fluids and solids

- Use constraints alike
  - principle of isotropy, homogeneity
  - principle of objectivity
  - principle of entropy production
Introduction - III

Goal:

Set a model problem

Find the primary fields

Measure the irreversibility over entropy production
Outline

• Introduction

• Setting a model problem to calculate the entropy production

• Outlook
Application - I

- Representation of second order tensor [Spencer 1971]

\[ \sigma_{ij} = \pi(\cdots)\delta_{ij} + 2\mu(\cdots)d_{ij} + h(\cdots)d_{ik}d_{kj} \quad \pi = \pi(d_{ll},d_{lm}d_{ml},d_{lm}d_{mn}d_{nl}) \]

\[ \mu = \mu(d_{ll},d_{lm}d_{ml},d_{lm}d_{mn}d_{nl}) \]

\[ h = h(d_{ll},d_{lm}d_{ml},d_{lm}d_{mn}d_{nl}) \]

- Channel flow, ansatz: \( \nu = (\nu_1(x_2), 0, 0) \) hence no vorticity \( \sigma_{ij} = \sigma_{ji} \)

\[ d_{ij} = \frac{1}{2}\left(\frac{\partial \nu_i}{\partial x_j} + \frac{\partial \nu_j}{\partial x_i}\right) \quad d_{ij} = \begin{pmatrix} 0 & \frac{1}{2} \frac{d \nu_1}{dx_2} & 0 \\ \frac{1}{2} \frac{d \nu_1}{dx_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ d_{ll} = 0 \quad , \quad d_{lm}d_{ml} = \frac{1}{2}\left(\frac{d \nu_1}{dx_2}\right)^2 \quad , \quad d_{lm}d_{mn}d_{nl} = 0 \]
• prefer to have constants on the diagonal

\[ \pi(\cdots) = -p \]

eliminate tensoriell-non-linear terms \[ h(\cdots) = 0 \]

• Viscous fluid and plastic deformable solid body (inspired from [Ziegler 1983])

\[ \sigma_{ij} = -p \delta_{ij} + 2 \left( \mu_0 + \frac{2k}{\pi \sqrt{d_{ij}d_{ij}}} \arctan \left( \frac{\sqrt{d_{ij}d_{ij}}}{2b} \right) \right) d_{ij} \]
Application - III

- Channel flow, ansatz: \( \mathbf{v} = (v_1(x_2), 0, 0) \)

\[
\sigma_{ij} = \begin{pmatrix}
-p & 2\mu_0d_{12} + \frac{2k}{\pi}\arctan\left(\frac{|d_{12}|}{b}\right)\text{sign}(d_{12}) & 0 \\
\vdots & -p & 0 \\
\vdots & \vdots & \vdots \\
\vdots & -p & \vdots 
\end{pmatrix}
\]

\( b \rightarrow 0 \)
Application - IV

• Setting the channel flow problem

• conservation laws

\[ \frac{d\rho}{dt} + \rho \frac{\partial u_i}{\partial x_i} = 0 \]

\[ \rho \frac{d\nu_i}{dt} + \frac{\partial \sigma_{ji}}{\partial x_j} = \rho f_i \]

\[ \rho \frac{du}{dt} + \rho \frac{\partial q_i}{\partial x_i} = \sigma_{ji} \frac{\partial \nu_i}{\partial x_j} + \rho r \]

• state laws

\[ \sigma_{12} = 2\alpha d_{12} + \frac{2}{\pi} \kappa \arctan \left( \frac{d_{12}}{b} \right) \]

\[ \nu = (\nu_1(x_2), 0, 0) \]

\[ u = u(x_2) \]

\[ T = T(x_2) \]

• semi-inverse ansatz

• assume

• steady state

\[ \frac{\partial}{\partial t} = 0 \]

• no body forces

\[ f_i = 0 \]

• no radiation supply

\[ r = 0 \]
Application - V

- boundary value non-linear differential equation

\[ p = p(x_1) \]

\[- \frac{d p}{d x_1} + \frac{d^2 \nu_1}{d x_2^2} \left( \mu_0 + \frac{4kb}{\pi (4b^2 + \left( \frac{d \nu_1}{d x_2} \right)^2)} \right) = 0 , \quad + \kappa \frac{d^2 T}{d x_2^2} + \mu_0 \left( \frac{d \nu_1}{d x_2} \right)^2 + \frac{2 \kappa}{\pi} \frac{d \nu_1}{d x_2} \arctan \left( \frac{d \nu_1}{d x_2} \right) = 0 \]

- numerical approximation is possible, by shooting method, finite differences

max. change on boundaries!

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Application - VI

- analytic solution for $b = 0$

\[
\bar{\sigma} = \frac{d \bar{v}_1}{d x_2} + k \text{ sign}\left(\frac{d \bar{v}_1}{d x_2}\right)
\]

\[
\bar{v}_1 = \frac{v_1}{v_0}, \quad v_0 = \left|\frac{d p}{d x_1}\right| \frac{R^2}{\mu_0}, \quad \bar{x} = \frac{x_2}{R},
\]

\[
\bar{v}_1 = \frac{1}{2} \left(1 - \bar{x}^2\right) - \xi (1 \pm \eta)(1 \mp \bar{x}) + \nu_{\text{Top/Bottom}}
\]

2nd order in $x$
Application - VII

• analytic solution for $b = 0$

$$+ \kappa \frac{d^2 T}{d x_2^2} + \sigma_{21} \frac{d \nu_1}{d x_2} = 0$$

$$\bar{T} = \frac{T}{T_{\text{boundary}}}, \quad \bar{\kappa} = \frac{\kappa T_{\text{boundary}}}{k \nu_0 R}, \quad \bar{p} = \left| \frac{d}{d x_1} \right| \frac{R}{k} , \quad T(\bar{x} = \pm 1) = T_{\text{boundary}}$$

4th order in $x$

$$\bar{T} = 1 + \frac{p}{12 \kappa} \left( -\bar{x}^4 + 1 + (\bar{x} + 1)4\xi^3 (\eta \pm 1)^3 \right) +$$

$$+ \frac{p}{6 \kappa} \left( 2\xi(\eta \pm 1) + \bar{k} \right) \left( \bar{x}^3 + 1 - (\bar{x} + 1)3\xi^2 (\eta \pm 1)^2 \right) +$$

$$+ \frac{p}{2 \kappa} \left( \xi^2 (1 \pm \eta)^2 - \bar{k} \xi(1 \pm \eta) \right) \left( -\bar{x}^2 + 1 + (\bar{x} + 1)2\xi(\eta \pm 1) \right)$$
Application - VIII

Assumption: Production of entropy is the measure of irreversibility

\[ \Sigma = \frac{\Sigma RT_{\text{boundary}}}{k \nu_0}, \]

\[ \Sigma T = \frac{\kappa}{T} \left( \frac{dT}{d\bar{x}} \right)^2 + \frac{1}{k} \left( \frac{d\bar{u}_1}{d\bar{x}} \right)^2 + \left( \frac{d\bar{u}_1}{d\bar{x}} \right) \]

Due to the balance of internal energy 2nd order in x  
Due to the balance of linear momentum 2nd order in x
• Assumption: Production of entropy is the measure of irreversibility
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Outlook

- Set a model problem with a more general ansatz
- Keep away from unstable regions (variable viscosity)
- Define a non-linear solution strategy
- Set a model problem with the complete stress representation
Thanks a lot for Your attention!

Questions?