Theorieaufgaben

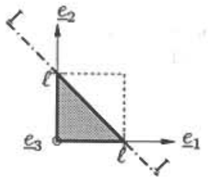
1. Gegeben ist der im dreidimensionalen Raum definierte Vektor $\underline{m} = \operatorname{div}\ (\underline{x} \times \underline{\underline{\sigma}}) = \frac{\partial}{\partial x_l}\left(\varepsilon_{ijk} x_j \sigma_{kl}\right)$. Differenzieren Sie den Ausdruck in einem ersten Schritt und spezialisieren Sie dann das Ergebnis auf die erste Komponente m_1. Achtung: Der Spannungstensor soll dabei nicht als symmetrisch angenommen werden!

Geg.: $\underline{x} = (x_1, x_2, x_3)$, $\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$

$m_i = \varepsilon_{ijk}\left(\frac{\partial x_j}{\partial x_l}\sigma_{kl} + x_j \frac{\partial \sigma_{kl}}{\partial x_l}\right) = \varepsilon_{ijk}\left(\sigma_{kj} + x_j \frac{\partial \sigma_{kl}}{\partial x_l}\right)$

$m_1 = \sigma_{32} + x_2\left(\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3}\right) - \left(\sigma_{23} + x_3\left(\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3}\right)\right)$

(2 Punkte)

2. 

Gegeben ist der 3-dimensionale Spannungszustand eines Würfels. Berechnen Sie den Spannungsvektor am Schnitt I-I gemäß der Cauchy-Tetraedergleichung.

Geg.: ℓ, $\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{23} & \sigma_{33} \end{bmatrix}$ $\quad n = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

(1 Punkt)

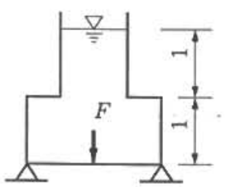
$\underset{\sim}{t} = \left(\frac{1}{\sqrt{2}}(\sigma_{11} + \sigma_{12}), \frac{1}{\sqrt{2}}(\sigma_{12} + \sigma_{22}), \frac{1}{\sqrt{2}}(\sigma_{31} + \sigma_{23})\right)$

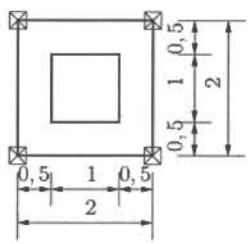
3. Gegeben ist das Feld $\psi(\underline{x}, t)$. Notieren Sie die substantielle Zeitableitung des Volumenintegrals, indem Sie das Reynoldssche Transporttheorem anwenden.

$$\frac{d}{dt}\int_{V(t)} \psi(\underline{x}, t)\, dV = \int_{V(t)} \frac{\partial \psi}{\partial t}\, dV + \oint_{\partial V(t)} \psi\, \underset{\sim}{v} \cdot \underset{\sim}{n}\, dA$$

(1 Punkt)

4. Schnitt: Draufsicht:





Welche resultierende Druckkraft F wird von dem im Behälter befindlichen Wasser auf den Behälterboden ausgeübt? Die Abmessungen des Behälters sind der Skizze zu entnehmen, alle Angaben sind in Meter.

Geg.: $\varrho = 1000\text{kg/m}^3$, $g = 10\text{m/s}^2$

F= $\rho g h A = 1000\ \text{kg/m}^3 \cdot 10\ \text{m/s}^2 \cdot 2\text{m} \cdot 4\text{m}^2$
$= 80.000\ \text{N} = 80\ \text{kN}$

(1 Punkt)

5. Geben Sie die [illegible]einheiten folgender Größen in den Einheiten 1, kg, m und s an:

spezifische Volumenkraft $\underline{f}$	m/s²
Dehnungstensor $\underline{\underline{\varepsilon}}$	1
Massenfluß $\dot{m}$	kg/s

(1 Punkt)

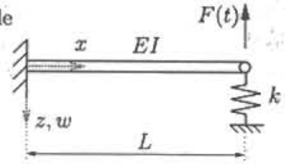
6. Beschreiben Sie kurz in Worten, worin der Unterschied der Eulerschen zur Lagrangeschen Betrachtungsweise zur Darstellung von Bilanzgleichungen in der Mechanik besteht.

Euler: Feldbetrachtung, festes Gitter

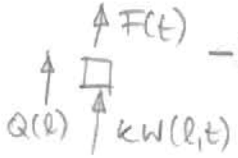
Lagrange: Masseteilchen wird verfolgt durch Einführung einer Referenzkonfiguration

(1 Punkt)

7. Wie lauten die zwei Randbedingungen am rechten Balkenende für den skizzierten transversal schwingenden Balken?



$M(x=\ell,t) = 0 \Rightarrow \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=\ell,t} = 0$



$-EI \left.\frac{\partial^3 w}{\partial x^3}\right|_{\ell,t} + kw(\ell,t) + F(t) = 0$

(2 Punkte)

8. Ist in einem runden Stab aus Stahl die Wellenausbreitungsgeschwindigkeit der Scherwellen (Torsion) c_T größer, kleiner oder gleich der Wellenausbreitungsgeschwindigkeit der Longitudinalwellen c_L? Kreuzen Sie die richtige Antwort bitte an.

- ○ $c_T > c_L$
- ⊗ $c_T < c_L$
- ○ $c_T = c_L$

(1 Punkt)

Aufg. 1

a) $\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$ mit $c^2 = \frac{N}{\rho A}$

b) $\frac{\partial^2 w}{\partial t^2} = X(x) \cdot \ddot{T}(t)$

$\frac{\partial^2 w}{\partial x^2} = X''(x) \cdot T(t)$

$\Rightarrow X(x) \cdot \ddot{T}(t) = c^2 X''(x) T(t)$

$\Rightarrow \frac{\ddot{T}(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} =: -\omega^2$

$\Rightarrow \ddot{T}(t) + \omega^2 T(t) = 0$

$X''(x) + \frac{\omega^2}{c^2} X(x) = 0$ mit $\frac{\omega^2}{c^2} =: \lambda^2$

c) $w(x=0,t) = 0$

$w(x=l,t) = 0$

d) $\Rightarrow X(0) = B \overset{!}{=} 0$

$X(l) = A \sin(\lambda l) = 0 \Rightarrow \sin \lambda l = 0$

$\lambda l = n\pi$

$\frac{\omega}{c} = \frac{n\pi}{l} \Rightarrow \omega_n = \frac{c n \pi}{l}$ mit $n = 1, 2, \ldots, \infty$

e) $w(x,t) = \sum_{n=1}^{\infty} (C_n \sin(\omega_n t) + D_n \cos(\omega_n t)) \sin(\lambda_n x)$

$w(x,t=0) = \sum_{n=1}^{\infty} D_n \sin(\lambda_n x) \overset{!}{=} 0 \Rightarrow D_n = 0$

$\frac{\partial w}{\partial t}\Big|_{x,t=0} = \sum_{n=1}^{\infty} C_n \omega_n \cos(\omega_n t) \sin(\lambda_n x)\Big|_{t=0} = \sum_{n=1}^{\infty} C_n \omega_n \sin(\lambda_n x)$

$\overset{!}{=} \begin{cases} v_0 \cos \frac{\pi(x-\xi)}{d} & \text{für } \xi - \frac{d}{2} \le x \le \xi + \frac{d}{2} \\ 0 & \text{sonst} \end{cases}$

noch Aufg. 1

$\int_0^l \sum_{n=1}^{\infty} C_n \omega_n \sin(\lambda_n x) \sin(\lambda_k x)\, dx = \int_{\xi - \frac{d}{2}}^{\xi + \frac{d}{2}} v_0 \cos \frac{\pi(x-\xi)}{d} \sin(\lambda_k x)\, dx$

$C_k \omega_k \frac{l}{2} = v_0 \frac{2 d \pi}{\pi^2 - \lambda_k^2} \cos\left(\frac{\lambda_k d}{2}\right) \sin(\lambda_k \xi)$

$\Rightarrow C_k = \frac{4 d \pi v_0}{l \omega_k (\pi^2 - \lambda_k^2)} \cos\left(\frac{\lambda_k d}{2}\right) \sin(\lambda_k \xi)$

$w(x,t) = \sum_{n=1}^{\infty} \frac{4 d \pi v_0}{l \omega_n (\pi^2 - \lambda_n^2)} \cos\left(\frac{\lambda_n d}{2}\right) \sin(\lambda_n \xi) \sin(\omega_n t) \sin(\lambda_n x)$

Aufg. 2

a) $\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$

b) $z_1 = x - ct \ , \ z_2 = x + ct$

c) $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial z_1}\frac{\partial z_1}{\partial t} + \frac{\partial w}{\partial z_2}\frac{\partial z_2}{\partial t} = c\left(\frac{\partial w}{\partial z_2} - \frac{\partial w}{\partial z_1}\right)$

$\frac{\partial^2 w}{\partial t^2} = c\left(\frac{\partial^2 w}{\partial z_1 \partial z_2}\frac{\partial z_1}{\partial t} + \frac{\partial^2 w}{\partial z_2^2}\frac{\partial z_2}{\partial t} - \frac{\partial^2 w}{\partial z_1^2}\frac{\partial z_1}{\partial t} - \frac{\partial^2 w}{\partial z_1 \partial z_2}\frac{\partial z_2}{\partial t}\right)$

$= c^2\left(-\frac{\partial^2 w}{\partial z_1 \partial z_2} + \frac{\partial^2 w}{\partial z_1^2} + \frac{\partial^2 w}{\partial z_2^2} - \frac{\partial^2 w}{\partial z_1 \partial z_2}\right)$

$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial z_1}\underbrace{\frac{\partial z_1}{\partial x}}_{=1} + \frac{\partial w}{\partial z_2}\underbrace{\frac{\partial z_2}{\partial x}}_{=1} = \frac{\partial w}{\partial z_1} + \frac{\partial w}{\partial z_2}$

$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial z_1^2} + \frac{\partial^2 w}{\partial z_2^2} + \frac{\partial^2 w}{\partial z_1 \partial z_2} + \frac{\partial^2 w}{\partial z_1 \partial z_2}$

$\Rightarrow \quad c^2\left(\frac{\partial^2 w}{\partial z_1^2} + \frac{\partial^2 w}{\partial z_2^2} - 2\frac{\partial^2 w}{\partial z_1 \partial z_2}\right) = c^2\left(\frac{\partial^2 w}{\partial z_1^2} + \frac{\partial^2 w}{\partial z_2^2} + 2\frac{\partial^2 w}{\partial z_1 \partial z_2}\right)$

$\Rightarrow \quad 4\frac{\partial^2 w}{\partial z_1 \partial z_2} = 0$

$\Rightarrow \int\limits_{(z_1)}\int\limits_{(z_2)} \frac{\partial^2 w}{\partial z_1 \partial z_2}\, dz_1 dz_2 = w(z_1, z_2) = \int\limits_{(z_1)}\int\limits_{(z_2)} 0 \, dz_1 dz_2$

$= f(z_1) + g(z_2)$

d) $\frac{\partial w}{\partial t} = \frac{\partial f}{\partial z_1}\frac{\partial z_1}{\partial t} + \frac{\partial g}{\partial z_2}\frac{\partial z_2}{\partial t} = c\left(-\frac{\partial f}{\partial z_1} + \frac{\partial g}{\partial z_2}\right)$

$\left.\frac{\partial w}{\partial t}\right|_{x, t=0} = c(-f' + g') = 0 \quad \Rightarrow \quad -f(x) + g(x) = \int_0^\ell 0\, dx = 0$

$w(x, t=0) = f(x) + g(x) = w_A(x)$

$\frac{(1)+(2)}{2} \Rightarrow \quad g(x) = \frac{1}{2}w_A(x) \quad$ bzw $\quad g(z_2) = \frac{1}{2}w_A(z_2)$

$\frac{(2)-(1)}{2} \Rightarrow \quad f(x) = \frac{1}{2}w_A(x) \quad$ bzw $\quad f(z_1) = \frac{1}{2}w_A(z_1)$

noch Aufg. 2

e) $w(x=0, t) = 0$

$w(x=\ell, t) = 0$

$w(0,t) = \frac{1}{2}w_0\left(\sin\left(\frac{\pi(0-ct)}{\ell}\right) + \sin\left(\frac{\pi(0+ct)}{\ell}\right)\right) = \frac{1}{2}w_0\left[\underbrace{\sin(0)}_{=0}\cos\left(\frac{ct}{\ell}\right) - \underbrace{\cos(0)}_{1}\sin\left(\frac{ct}{\ell}\right)\right.$

$\left. + \underbrace{\sin(0)}_{=0}\cos\left(\frac{ct}{\ell}\right) + \underbrace{\cos(0)}_{1}\sin\left(\frac{ct}{\ell}\right)\right] = 0$

$w(\ell,t) = \frac{1}{2}w_0\left(\sin\left(\frac{\pi(\ell-ct)}{\ell}\right) + \sin\left(\frac{\pi(\ell+ct)}{\ell}\right)\right) = \frac{1}{2}w_0\left[\underbrace{\sin(\pi)}_{=0}\cos\left(\frac{\pi ct}{\ell}\right) - \underbrace{\cos(\pi)}_{-1}\sin\left(\frac{\pi ct}{\ell}\right)\right.$

$\left. + \underbrace{\sin(\pi)}_{=0}\cos\left(\frac{\pi ct}{\ell}\right) + \underbrace{\cos(\pi)}_{-1}\sin\left(\frac{\pi ct}{\ell}\right)\right]$

$= 0$

Aufg. 3

a) $\frac{\partial \rho v_i}{\partial t} + \frac{\partial(\rho v_i v_j)}{\partial x_j} = \rho f_i + \frac{\partial \sigma_{ij}}{\partial x_j}$

oder: $\frac{d}{dt}(\rho v_i) + \rho v_i \frac{\partial v_j}{\partial x_j} = \rho f_i + \frac{\partial \sigma_{ij}}{\partial x_j}$

oder: $\rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = \rho f_i + \frac{\partial \sigma_{ij}}{\partial x_j}$

b) $\sigma_{ij} = -p\delta_{ij}$ 1

$f = (0,0,g)$ 1

$\Rightarrow 0 = \frac{\partial p}{\partial x_1} \Rightarrow p(x_1) = \text{const} \Rightarrow p = p(x_2, x_3)$

$0 = \frac{\partial p}{\partial x_2} \Rightarrow p(x_2) = \text{const} \Rightarrow p = p(x_3)$

$\rho g = \frac{\partial p}{\partial x_3} \overset{*}{=} \frac{dp}{dx_3}$

c) $\frac{dp}{dx_3} = \frac{\partial p}{\partial \rho}\frac{d\rho}{dx_3} + \frac{\partial p}{\partial T}\frac{dT}{\partial x_3} = \frac{\partial p}{\partial \rho}\frac{d\rho}{dx_3}$

$\kappa_T \overset{\text{def}}{=} -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$ mit $V = \frac{m}{\rho}$ folgt $\kappa_T = -\frac{\rho}{m}\left(\frac{\partial(\frac{m}{\rho})}{\partial p}\right)$

wegen $m = \text{const}$ $= -\rho\left(\frac{\partial(\frac{1}{\rho})}{\partial p}\right)_T$

$= -\rho\left(-\frac{1}{\rho^2}\frac{\partial \rho}{\partial p}\right)_T$

$= \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T$

$\Rightarrow \kappa_T \rho = \left(\frac{\partial \rho}{\partial p}\right)_T$

$\Rightarrow \frac{dp}{dx_3} = \frac{1}{\kappa_T \rho}\frac{d\rho}{dx_3}$

noch Aufg. 3

d) $\rho g = \frac{1}{\kappa_T \rho}\frac{d\rho}{dx_3}$

$\Rightarrow \kappa_T g\, dx_3 = \frac{1}{\rho^2} d\rho$

$\Rightarrow \int_0^z \kappa_T g\, dx_3 = \int_{\rho_0}^{\rho(z)} \frac{1}{\rho^2}\, d\rho$

$\kappa_T g z = \left[-\frac{1}{\rho}\right]_{\rho_0}^{\rho(z)} = -\frac{1}{\rho(z)} + \frac{1}{\rho_0}$

$\Rightarrow \rho(z) = \left[\frac{1}{\rho_0} - \kappa_T g z\right]^{-1}$

e) $\frac{dp}{dx_3} = \frac{1}{\rho \kappa_T}\frac{d\rho}{dx_3}$ bzw. $dp = \frac{1}{\rho \kappa_T} d\rho$

$\int_{p_0}^{p(z)} dp = \frac{1}{\kappa_T}\int_{\rho_0}^{\rho(z)} \frac{1}{\rho} d\rho$

$p(z) - p_0 = \frac{1}{\kappa_T}\left[\ln \rho\right]_{\rho_0}^{\rho(z)} = \frac{1}{\kappa_T}\ln\left(\frac{\rho(z)}{\rho_0}\right)$

$p(z) = p_0 + \frac{1}{\kappa_T}\ln\left(\rho_0\left(\frac{1}{\rho_0} - \kappa_T g z\right)\right)^{-1}$

$= p_0 + \frac{1}{\kappa_T}\ln\left(1 - \kappa_T \rho_0 g z\right)^{-1}$

oder: $\int_{p_0}^{p(z)} dp = \int_0^z \rho g\, dx_3 = \int_0^z \left(\frac{1}{\rho_0} - \kappa_T g x_3\right)^{-1} g\, dx_3$

Subst. $\frac{1}{\rho_0} - \kappa_T g x_3 =: A \Rightarrow \frac{dA}{dx} = -\kappa_T g \Rightarrow dx = -\frac{1}{\kappa_T g} dA$

$\Rightarrow -\frac{1}{\kappa_T g}\int_{A=\frac{1}{\rho_0}}^{A=\frac{1}{\rho_0}-\kappa_T g z} \frac{1}{A}\, dA = -\frac{1}{\kappa_T g}\left[\ln A\right]_{\frac{1}{\rho_0}}^{\frac{1}{\rho_0}-\kappa_T g z} = -\frac{1}{\kappa_T g}\ln\left(\frac{\frac{1}{\rho_0} - \kappa_T g z}{\frac{1}{\rho_0}}\right)$ 1

$p - p_0 = \frac{\cancel{g}}{\kappa_T \cancel{g}}\ln\left(\frac{\frac{1}{\rho_0}}{\frac{1}{\rho_0} - \kappa_T g z}\right) = \frac{1}{\kappa_T}\ln\left(\frac{1}{\rho_0\left(\frac{1}{\rho_0} - \kappa_T g z\right)}\right)$

$p = p_0 + \frac{1}{\kappa_T}\ln\left(1 - \kappa_T \rho_0 g z\right)^{-1}$

noch Aufg 3

$$\lim_{k_T \to 0} \frac{\ln\left(\frac{1}{1-k_T \rho_0 g z}\right)}{k_T} = \lim_{k_T \to 0} \frac{\frac{1}{\frac{1}{1-k_T \rho g z}} \cdot \left(-(1-k_T \rho g z)^{-2}\right)(-\rho_0 g z)}{1}$$

$$= \frac{1 \cdot (-1) \cdot (-\rho_0 g z)}{1} = \rho_0 g z$$