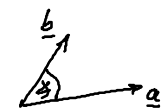
Vorlesung am 24.5.2018

- Wiederholung Energiesatz und Beispiele
- Drehimpulssatz
- Kinetik und Kinematik von MP-Systemen

Wiederholung

$$\underline{a}\cdot\underline{b} = |\underline{a}||\underline{b}|\cos\sphericalangle$$



$$\text{dim } \underline{F}\cdot\underline{v} = N\frac{m}{s} = \frac{\text{J}}{s} = \text{W}$$

$$\text{Leistung} = \frac{\text{Energie}}{\text{Zeit}}$$

Newton $\quad m\frac{d\underline{v}}{dt} = \underline{F} \quad \Big| \cdot \underline{v}$

$$\frac{d}{dt}\Big(m\underbrace{\frac{\underline{v}\cdot\underline{v}}{2}}_{\underline{v}^2}\Big) = \underline{F}\cdot\underbrace{\underline{v}}_{\frac{d\underline{x}}{dt}} \equiv P \quad , \quad E^{kin}(t) := \frac{m}{2}\underline{v}(t)\cdot\underline{v}(t) = \frac{m}{2}\underline{v}^2$$

$$\frac{dE_{kin}(t)}{dt} = P(t) \quad \Big| \int_{\tilde t=0}^{\tilde t=t} -''- \, d\tilde t$$

$$E_{kin}(t) - E_{kin}(0) = \int_{\tilde t=0}^{\tilde t=t} P(\tilde t)\, d\tilde t = \int_{\tilde t=0}^{\tilde t=t} \underline{F}\cdot\frac{d\underline{x}}{d\tilde t}\, d\tilde t = \int_{\tilde{\underline{x}}=\underline{x}_0}^{\tilde{\underline{x}}=\underline{x}} \underline{F}\cdot d\tilde{\underline{x}} =$$

$$= \int_{\tilde{\underline{x}}=\underline{x}_0}^{\tilde{\underline{x}}=\underline{x}} \underline{F}^{pot}\cdot d\tilde{\underline{x}} + \int_{\tilde{\underline{x}}=\underline{x}_0}^{\tilde{\underline{x}}=\underline{x}} \underline{F}^{diss}\cdot d\underline{x} = -\int_{\underline{x}_0}^{\underline{x}} dE^{pot}(\tilde{\underline{x}}) + \int_{\tilde{\underline{x}}=\underline{x}_0}^{\tilde{\underline{x}}=\underline{x}} \underline{F}^{diss}\cdot d\tilde{\underline{x}} =$$

$$\underline{F} = \underline{F}^{pot} + \underline{F}^{diss}$$

$$= -\left[E^{pot}(\underline{x}) - E^{pot}(\underline{x}_0)\right] + \int_{\tilde{\underline{x}}=\underline{x}_0}^{\underline{x}} \underline{F}^{diss}\cdot d\tilde{\underline{x}}$$

Bei Potentialkräften

$$\underline{F}^{pot} = -\underline{\nabla}E^{pot}$$

$$= -\left[\underline{e}_1\frac{\partial E^{pot}}{\partial x_1} + \underline{e}_2\frac{\partial E^{pot}}{\partial x_2} + \underline{e}_3\frac{\partial E^{pot}}{\partial x_3}\right]$$

Nabla-Operator

$$\underline{\nabla}(\cdot) = \underline{e}_1\frac{\partial(\cdot)}{\partial x_1} + \underline{e}_2\frac{\partial(\cdot)}{\partial x_2} + \underline{e}_3\frac{\partial(\cdot)}{\partial x_3}$$

wie bei einem Vektor

$$\underline{F} = F_1\underline{e}_1 + F_2\underline{e}_2 + F_3\underline{e}_3$$

$$\underline{F}^{pot}\cdot d\underline{x} = -\left[\underline{e}_1\frac{\partial E^{pot}}{\partial x_1} + \underline{e}_2\frac{\partial E^{pot}}{\partial x_2} + \underline{e}_3\frac{\partial E^{pot}}{\partial x_3}\right]\cdot\left[\underline{e}_1 dx_1 + \underline{e}_2 dx_2 + \underline{e}_3 dx_3\right] =$$

$$= -\left[\frac{\partial E^{pot}}{\partial x_1}dx_1 + \frac{\partial E^{pot}}{\partial x_2}dx_2 + \frac{\partial E^{pot}}{\partial x_3}dx_3\right] = -dE^{pot}(\underbrace{x_1, x_2, x_3}_{\underline{x}})$$

totales Differential

$$\Rightarrow \quad E^{kin}(t) + E^{pot}(t) = E^{kin}(0) + E^{pot}(0) + \underbrace{\int_{\hat{x}=x_0}^{\hat{x}=x} \underline{F}^{diss} \cdot d\underline{\tilde{x}}}_{W^{diss}}$$

Arbeitssatz!

Bei Abwesenheit von Dissipation:

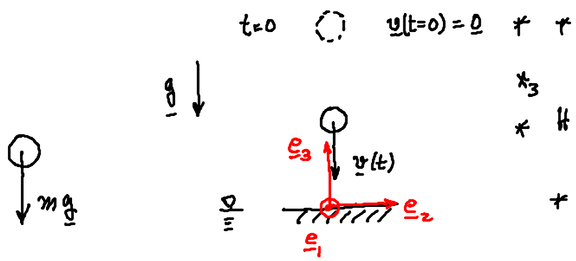
$$E^{kin}(t) + E^{pot}(t) = const = E^{kin}(0) + E^{pot}(0)$$

Beispiel

(a) freier Fall

Ges.: $\underline{v}(t) = ?$

Geg.: H, $\underline{v}(t=0) = \underline{0}$



$$E^{kin}(t) + E^{pot}(t) = E^{kin}(0) + E^{pot}(0), \quad W^{diss} = 0$$

$$\Downarrow$$

$$\left.\begin{aligned} E^{kin}(0) &= 0 \\ E^{kin}(t) &= \frac{m}{2}\, \underline{v}(t) \cdot \underline{v}(t) \\ E^{pot}(t) &= mg\,(H - x_3) \\ E^{pot}(0) &= mgH \end{aligned}\right\} \quad \frac{\cancel{m}}{2}|\underline{v}(t)|^2 + \cancel{m}g(\cancel{H} - x_3) = \cancel{mgH}$$

$$\Rightarrow |\underline{v}(t)| = \sqrt{2gx_3}$$

$$\underline{v}(t) = -v_3(t)\,\underline{e}_3$$

$$|\underline{v}(t)| = v_3(t)\underbrace{|\underline{e}_3|}_{1}$$

$$\Rightarrow v_3(t) = \sqrt{2gx_3}$$

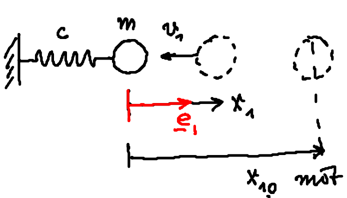
$$\underline{F}^{pot} \equiv \underline{F}^{grav} = m\underline{g} = -mg\,\underline{e}_3$$

$$= -\underline{\nabla}\, E^{pot} = -\left(\underline{e}_1 \frac{\partial}{\partial x_1} + \underline{e}_2 \frac{\partial}{\partial x_2} + \underline{e}_3 \frac{\partial}{\partial x_3}\right)(mg\,x_3 + C) = -\underline{e}_1\,0 - \underline{e}_2 \cdot 0 - \underline{e}_3\, mg$$

$$E^{pot} = mg\,x_3 + C$$

(b) Hookesche Feder

Ziel: Berechnung von $v_1(t)$ mit dem Energiesatz



$$E^{kin}(t) + E^{pot}(t) = E^{kin}(0) + E^{pot}(0), \quad W^{diss} = 0$$

$$\Downarrow$$

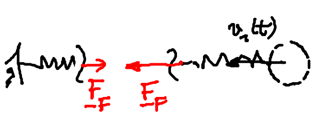
$$\left.\begin{aligned} E^{kin}(t) &= \frac{m}{2}\, \underline{v}(t) \cdot \underline{v}(t) = \frac{m}{2} v_1^2(t) \\ & \qquad \underline{v}(t) = -v_1(t)\,\underline{e}_1 \\ E^{pot}(t) &= \frac{c}{2} x_1^2 \\ E^{pot}(0) &= \frac{c}{2} x_{1,0}^2 \end{aligned}\right\} \Rightarrow \frac{m}{2} v_1^2(t) + \frac{c}{2} x_1^2 = \frac{c}{2} x_{1,0}^2$$

$$\Rightarrow v_1 = \sqrt{\frac{c}{m}\left(x_{1,0}^2 - x_1^2\right)}$$

$$\underline{F}^{pot} = -\underline{\nabla}\, E^{pot} = -\left(\underline{e}_1 \frac{\partial}{\partial x_1} + \underline{e}_2 \frac{\partial}{\partial x_2} + \underline{e}_3 \frac{\partial}{\partial x_3}\right)\frac{c}{2} x_1^2 = -\underline{e}_1\, c\, x_1$$

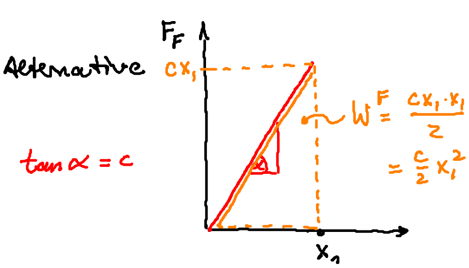
$$E^{pot} = \frac{c}{2} x_1^2$$

Freischnitt



$$\underline{F}_F = -F_F\,\underline{e}_1 = -c\,x_1\,\underline{e}_1$$

Alternative

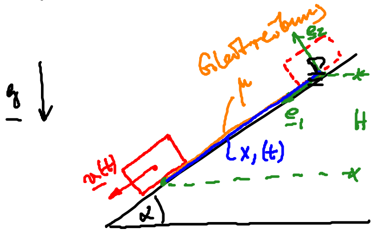


$$W^F = \int_{\tilde{x}_1=0}^{\tilde{x}_1=x_1} \underbrace{c\,\tilde{x}_1}_{F^F}\, d\tilde{x}_1 = \frac{c}{2}\, \tilde{x}_1^2 \Big|_{\tilde{x}_1=0}^{\tilde{x}_1=x_1} = \frac{c}{2} x_1^2 \equiv E^{pot}$$

(c) Rutschen auf schiefer Ebene

Gesucht $\underline{v}(t) = \underline{\underline{v_1(t)}}\,\underline{e}_1$

Gegeben: $\mu,\ \alpha,\ x_1(t)$

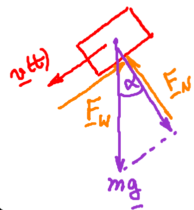


Arbeitssatz

$$E^{kin}(t) + E^{pot}(t) = E^{kin}(0) + E^{pot}(0) + W^{diss}$$

$$\Downarrow$$

$$\left.\begin{aligned} E^{kin}(t) &= \frac{m}{2} v_1^2(t) \\ E^{pot}(t) &= -mg\,H = -mg\sin\alpha\, x_1 \\ E^{kin}(0) &= 0 \qquad H = x_1 \sin\alpha \\ E^{pot}(0) &= 0 \end{aligned}\right\} \Rightarrow \quad \frac{m}{2} v_1^2(t) - mg\sin\alpha\, x_1 = -\mu m g \cos\alpha\, x_1$$



$F_W = \mu F_N$

$= \mu\, mg \cos\alpha$

$$W^{diss} = \int_{\tilde{x}=0}^{\tilde{x}=x(t)} \underline{F}^{diss} \cdot d\tilde{\underline{x}} =$$

$$\underline{F}_W = -F_W\,\underline{e}_1 = -\mu m g\cos\alpha\,\underline{e}_1$$

$$= \int_{\tilde{x}=0}^{\tilde{x}=x(t)} \underline{F}_W \cdot d\tilde{\underline{x}} + \underbrace{\int_{\tilde{x}=0}^{\tilde{x}=x(t)} \underline{F}_N \cdot d\tilde{\underline{x}}}_{} \qquad = -\mu\, mg\cos\alpha \int_{\tilde{x}=0}^{\tilde{x}=x(t)} \underline{e}_1 \cdot d\tilde{\underline{x}} = -\mu\, mg\cos\alpha \int_{\tilde{x}_1=0}^{\tilde{x}_1=x_1(t)} 1\, d\tilde{x}_1 =$$

$d\tilde{\underline{x}} \perp \underline{F}_N \Rightarrow d\tilde{\underline{x}} \cdot \underline{F}_N = 0$ $\qquad d\tilde{\underline{x}} = dx_1\,\underline{e}_1 \qquad \underline{e}_1 \cdot \underline{e}_1$

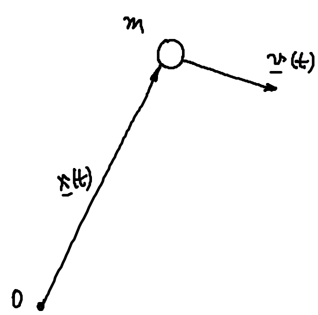
$$= -\mu m g\cos\alpha\, x_1(t)$$

$$\Rightarrow \quad v_1(t) = \sqrt{2g\,\underbrace{(\sin\alpha - \mu\cos\alpha)\,x_1}_{H}}$$

3.1.5 Drehimpuls und Drallsatz

Erinnere: Impuls $\quad \underline{p} := m\,\underline{v}$ $\qquad$ Newton $\quad \frac{d\underline{p}}{dt} = \underline{F}$

$$m\frac{d\underline{v}}{dt} = \underline{F} \qquad (1)$$



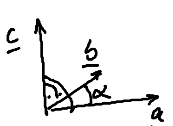
Nun Drehimpuls $\quad \underline{L}^{(0)} := \underline{x} \times \underline{p} = m\,\underline{x} \times \underline{v}$

Jetzt zeitliche Änderung von $\underline{L}^{(0)}$ $\qquad$ Kraft $\underline{F}$ $\qquad$ Moment $\underline{M}^{(0)} = \underline{x} \times \underline{F}$ (2)

$$\frac{d\underline{L}^{(0)}}{dt} = \frac{d}{dt}\left(m\,\underline{x}(t) \times \underline{v}(t)\right) = m\left(\overbrace{\frac{d\underline{x}(t)}{dt}}^{\underline{v}(t)} \times \underline{v}(t) + \underline{x}(t) \times \frac{d\underline{v}(t)}{dt}\right) =$$

Produktregel

$\underline{c} = \underline{a} \times \underline{b}$



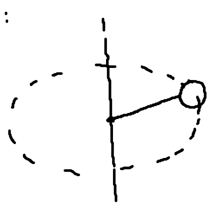
$|\underline{c}| = |\underline{a}|\,|\underline{b}|\sin\alpha$

$$= m\left(\underline{0} + \underline{x} \times \frac{d\underline{v}}{dt}\right) = \underline{x} \times \left(m\frac{d\underline{v}}{dt}\right) = \underline{x} \times \underline{F} = \underline{M}^{(0)}$$

Newton (1) $\qquad$ (2)

$$\Rightarrow \quad \frac{d\underline{L}^{(0)}}{dt} = \underline{M}^{(0)} \qquad \underset{\underline{\underline{M^{(0)}}}}{\Rightarrow} \qquad \underline{L}^{(0)}(t) = \underline{L}^{(0)}(0)$$

Falls keine Kräfte wirken:



3.2 Dynamik von MP-Systemen

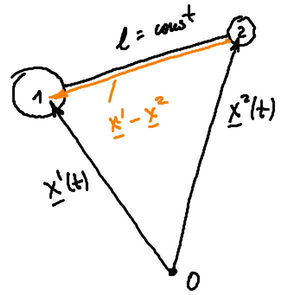
3.2.1 Kinematik

Problem: Beschreibung der Bindungen zwischen MP

$\Rightarrow$ kinematische Zwänge, Nebenbedingungen

Beispiel

(a) Hantel



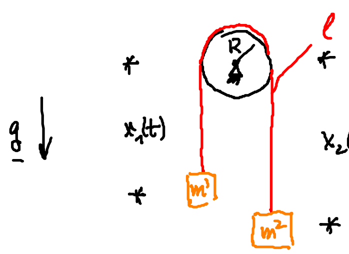
Nebenbedingung

$$(\underline{x}^1 - \underline{x}^2)\cdot(\underline{x}^1 - \underline{x}^2) \overset{!}{=} l^2 \quad \Big|\ \frac{d}{dt}$$

$$(\dot{\underline{x}}^1 - \dot{\underline{x}}^2)\cdot(\underline{x}^1 - \underline{x}^2) + (\underline{x}^1 - \underline{x}^2)\cdot(\dot{\underline{x}}^1 - \dot{\underline{x}}^2) = 0$$

$$\Rightarrow 2\underbrace{(\dot{\underline{x}}^1 - \dot{\underline{x}}^2)}_{\text{relative Geschw.}}\bullet\underbrace{(\underline{x}^1 - \underline{x}^2)}_{\text{Distanzvektor}} = 0$$

(b) Umlenkrolle



$$x_1(t) + x_2(t) + \pi R = l \quad \Big|\ \frac{d}{dt}$$

$$\dot{x}_1(t) + \dot{x}_2(t) = 0$$

$$\Rightarrow dx_1(t) = -dx_2(t)$$

Bewegung muß so sein, daß rel. Geschwindigkeitsvektor und Distanzvektor aufeinander senkrecht stehen