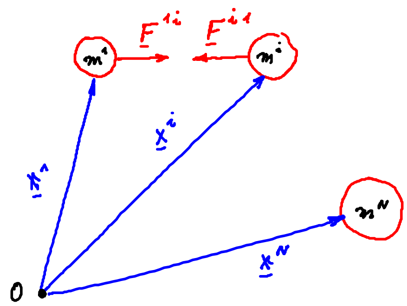
Vorlesung 28.5.2018

- Wiederholung Drehimpulssatz für MP-Systeme
- Beispiele
- Energie- und Arbeitssatz für MP-Systeme
- Beispiel: zentrischer Stoß zw. 2 MPs
- Körper mit zeitveränderlicher Masse

Wiederholung



$i = 1, \dots, N$

Newton

$$m^i \ddot{\underline{x}}^i = \underline{F}^i + \sum_j \underline{F}^{ij}$$

$$\underline{p}^i = m^i \dot{\underline{x}}^i$$

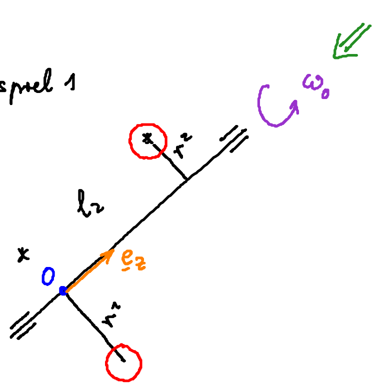
Drehimpuls

$$\underline{L}^{0,i} := \underline{x}^i \times \underline{p}^i = \underline{x}^i \times (m^i \dot{\underline{x}}^i) \quad \Rightarrow \quad \underline{L}^{(0)} = \sum_i \underline{L}^{0,i}$$

$$\frac{d\underline{L}^{0,i}}{dt} = \dot{\underline{x}}^i \times (m^i \dot{\underline{x}}^i) + \underline{x} \times (m^i \ddot{\underline{x}}^i) = \underline{x}^i \times \Big(\underline{F}^i + \sum_j \underline{F}^{ij} \Big) \quad , \quad \underline{M}^{0,i} = \underline{x}^i \times \underline{F}^i$$

$$\frac{d\underline{L}^{(0)}}{dt} = \underbrace{\sum_i \underline{M}^{0,i}}_{\underline{M}^0} + \underbrace{\sum_i \sum_j \underline{x}^i \times \underline{F}^{ij}}_{\underline{0} \text{ falls Zentralwechselwirkung}} \quad \Rightarrow \quad \frac{d\underline{L}^{(0)}}{dt} = \underline{M}^{(0)}$$

Beispiel 1



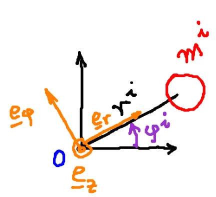
$$\underline{L}^{(0)} = \underline{x}^1 \times (m^1 \dot{\underline{x}}^1) + \underline{x}^2 \times (m^2 \dot{\underline{x}}^2)$$

$$= \big(r^1 \underline{e}_r(t)\big) \times \big(m^1 r^1 \omega_0 \underline{e}_\varphi(t)\big) +$$

$$+ \big(r^2 \underline{e}_r(t)\big) \times \big(m^2 r^2 \omega_0 \underline{e}_\varphi(t)\big) + \big(l_2 \underline{e}_z\big) \times r^2 \omega_0 \underline{e}_\varphi(t)$$

$$= m^1 (r^1)^2 \omega_0 \underbrace{\underline{e}_r(t) \times \underline{e}_\varphi(t)}_{\underline{e}_z} + m^2 (r^2)^2 \omega_0 \underline{e}_z +$$

$$+ m^2 r^2 l_2 \underbrace{\underline{e}_z \times \underline{e}_\varphi(t)}_{-\underline{e}_r(t)}$$



$$\underline{x}^1 = r^1 \underline{e}_r(t) \quad , \quad \underline{x}^2 = r^2 \underline{e}_r(t) + l_2 \underline{e}_z \quad , \quad \frac{d\underline{e}_r(t)}{dt} = \omega(t)\, \underline{e}_\varphi(t)$$

$$\dot{\underline{x}}_1 = r^1 \omega_0 \underline{e}_\varphi(t) \quad , \quad \dot{\underline{x}}^2 = r^2 \omega_0 \underline{e}_\varphi(t)$$

$$\Rightarrow \quad L_3^{(0)} = \left(m^1 (r^1)^2 + m^2 (r^2)^2\right)\omega_0 \qquad , \qquad \frac{d\varphi}{dt} = \omega(t) = \omega_0$$

$$= \left(m^1 (r^1)^2 + m^2 (r^2)^2\right)\frac{d\varphi}{dt}$$

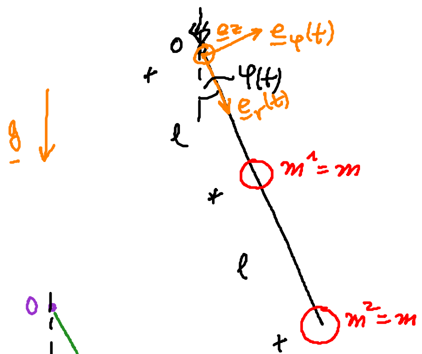
<u>Analogie:</u> 1 D – Bewegung

$$m\,\frac{d^2x}{dt^2} = F \qquad , \qquad \frac{dL_3^{(0)}}{dt} = M_3^{(0)}$$

$$\underbrace{\left(m^1 (r^1)^2 + m^2 (r^2)^2\right)}\,\frac{d^2\varphi}{dt^2} = M_3^{(0)}$$

Massenträgheitsmoment $\theta^{a-a} := m^1 (r^1)^2 + m^2 (r^2)^2$

Beispiel 2



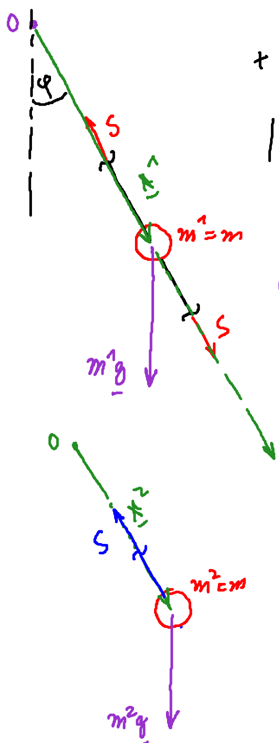
$$\frac{d\underline{L}^{(0)}}{dt} = \underline{M}^{(0)} \qquad , \qquad \begin{aligned} \underline{x}^1 &= \ell\,\underline{e}_r(t)\ , & \dot{\underline{x}}^1 &= \ell\,\dot\varphi(t)\,\underline{e}_\varphi(t) \\ \underline{x}^2 &= 2\ell\,\underline{e}_r(t)\ , & \dot{\underline{x}}^2 &= 2\ell\,\dot\varphi\,\underline{e}_\varphi(t) \end{aligned}$$

$$\underline{L}^{(0)} = \underline{x}^1 \times (\overset{m}{\overset{\parallel}{m^1}}\,\dot{\underline{x}}^1) + \underline{x}^2 \times (\overset{m}{\overset{\parallel}{m^2}}\,\dot{\underline{x}}^2) =$$

$$= m\,\underline{x}^1 \times \dot{\underline{x}}^1 + m\,\underline{x}^2 \times \dot{\underline{x}}^2 =$$

$$= m\,\ell^2\dot\varphi\,\underline{e}_r(t) \times \underline{e}_\varphi(t) + m\,(2\ell)^2\dot\varphi\,\underline{e}_r(t) \times \underline{e}_\varphi(t)$$

$$= (m\ell^2 + 4m\ell^2)\,\dot\varphi\,\underbrace{\underline{e}_r(t) \times \underline{e}_\varphi(t)}_{\underline{e}_z}$$



$$|\underline{a} \times \underline{b}| = |\underline{a}|\,|\underline{b}|\,\sin\sphericalangle$$

$$\underline{L}^{(0)}(t) = 5m\ell^2\dot\varphi(t)\,\underline{e}_z \quad \Rightarrow \quad \frac{d\underline{L}^{(0)}}{dt} = 5m\ell^2\ddot\varphi(t)\,\underline{e}_z$$

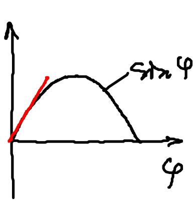
$$\left.\begin{aligned} \underline{M}^{0,1} &= \underline{x}^1 \times (m^1\underline{g}) \\ &= -m\ell g \sin\varphi\,\underline{e}_3 \\[2ex] \underline{M}^{0,2} &= \underline{x}^2 \times (m^2\underline{g}) \\ &= -2m\ell g \sin\varphi\,\underline{e}_3 \end{aligned}\right\}$$

$$\Downarrow$$

$$\frac{d\underline{L}^{(0)}}{dt} = \underline{M}^{(0)}$$

$$\Rightarrow \quad 5m\ell^2\ddot\varphi(t)\,\underline{e}_z = -\underbrace{(m\ell g\sin\varphi + 2m\ell g\sin\varphi)}_{3m\ell g\sin\varphi}\,\underline{e}_3$$

$$\Rightarrow \quad \ddot\varphi(t) + \frac{3}{5}\frac{g}{\ell}\sin\varphi = 0$$



für kleine φ : $\boxed{\ddot\varphi(t) + \underbrace{\frac{3}{5}\frac{g}{\ell}}_{\omega_0^2}\varphi = 0}$, $\sin\varphi \approx \varphi$

$$\dim\frac{g}{\ell} = \frac{m}{s^2}\,\frac{1}{m} = \frac{1}{s^2} \qquad \Rightarrow \qquad \omega_0 = \frac{2\pi}{T}$$

<u>3.2.5 Energie- und Arbeitssatz für MP-Systeme</u>

Starten von $m^i \ddot{\underline{x}}^i = \underline{F}^i + \sum_j \underline{F}^{ij}$ $\quad \Big| \cdot \dot{\underline{x}}^i \quad \Big| \sum_i$

$\underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a}$

$\underline{a}\times\underline{b} = -\underline{b}\times\underline{a}$

$$\Rightarrow \frac{d}{dt}\Big(\overbrace{\frac{m^i}{2}\, \dot{\underline{x}}^i\cdot\dot{\underline{x}}^i}^{E^{kin,i}}\Big) = \underbrace{\underline{F}^i\cdot\dot{\underline{x}}^i}_{P^i} + \sum_j \underline{F}^{ij}\cdot\dot{\underline{x}}^i$$

$\cdot \dot{\underline{x}}^i$

$$\frac{m^i}{2}\left(\ddot{\underline{x}}^i\cdot\dot{\underline{x}}^i + \dot{\underline{x}}^i\cdot\ddot{\underline{x}}^i\right) = \frac{m^i}{2}\left(\ddot{\underline{x}}^i\cdot\dot{\underline{x}}^i + \ddot{\underline{x}}^i\cdot\dot{\underline{x}}\right) = m^i\ddot{\underline{x}}^i\cdot\dot{\underline{x}}^i$$

$\sum_i$: $\quad \frac{d}{dt}\sum_i E^{kin,i} = \sum_i P^i + \sum_i\sum_j \underline{F}^{ij}\cdot\dot{\underline{x}}^i$

$E^{kin} = \sum_i E^{kin,i}$

$P = \sum_i P^i$

$$\boxed{\frac{d}{dt}E^{kin} = P} \longrightarrow \text{wie bei einem einzelnen MP}$$

Der verbleibende Summenterm ist dann Null, falls

a) Zentralkräfte vorliegen

b) starre Bindungen

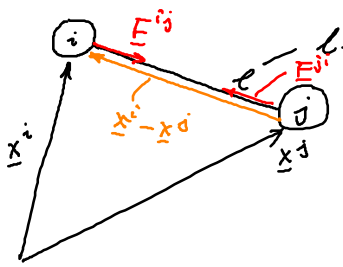
Beweis:

Binde

$$(\underline{x}^i - \underline{x}^j)\cdot(\underline{x}^i - \underline{x}^j) = \ell^2$$

$$\Rightarrow \cancel{2}\left(\dot{\underline{x}}^i - \dot{\underline{x}}^j\right)\cdot\left(\underline{x}^i - \underline{x}^j\right) = 0$$

$\frac{d}{dt}$



$$\underline{F}^{ij} = -\underline{F}^{ji}$$

$$\underline{F}^{ij}\cdot\dot{\underline{x}}^i + \underline{F}^{ji}\cdot\dot{\underline{x}}^j = \underline{F}^{ij}\cdot\left(\dot{\underline{x}}^i - \dot{\underline{x}}^j\right) = -|\underline{F}^{ij}|\,\frac{\left(\underline{x}^i - \underline{x}^j\right)\cdot\left(\dot{\underline{x}}^i - \dot{\underline{x}}^j\right)}{|\underline{x}^i - \underline{x}^j|} = 0$$

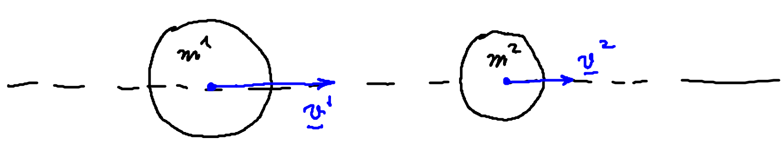
0 wegen starrer Bindung

Zentralkraft

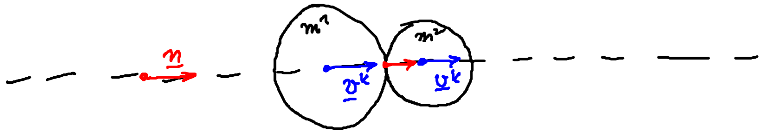
$$\underline{F}^{ij} = -|\underline{F}^{ij}|\,\frac{\underline{x}^i - \underline{x}^j}{|\underline{x}^i - \underline{x}^j|}$$

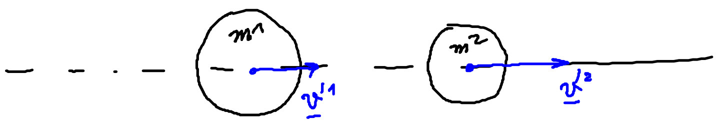
3.2.6 Zentrische Stöße von kugelförmigen Massen

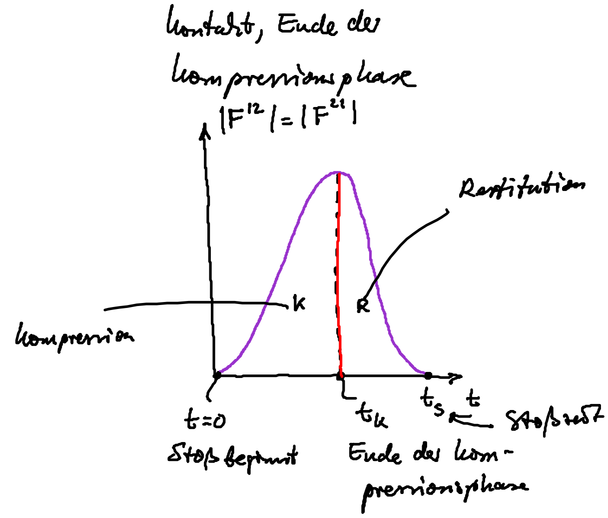
Beispiel zum Energie- und Impulssatz von MP-Systemen



kurz vor dem Stoß







Impulssatz zwischen Zeit 0 und t_k

für Masse 1 für Masse 2

$$(2)\quad m^1 v^k - m^1 v^1 = -k^k \,, \quad m^2 v^k - m^2 v^2 = k^k \quad (3)$$

Impulssatz zwischen Zeit t_k und t_s

für Masse 1 für Masse

$$(4)\quad m^1 v'^1 - m^1 v^k = -k^R \,, \quad m^2 v'^2 - m^2 v^k = k^R \quad (5)$$

5 Unbekannte $v'^1, v'^2, k^k, k^R, v^k$, 5 Gleichungen

Erinnere Stoßparameter e

plastisch

rein elastischer Stoß

$$0 \le e := \frac{k^R}{k^k} \le 1 \quad (1)$$

$$k^k = \int_{t=0}^{t=t_k} |F^{12}|\, dt \qquad \text{Kraftstoß Kompression}$$

$$k^R = \int_{t=t_k}^{t=t_s} |F^{12}|\, dt \qquad \text{Kraftstoß Restitution}$$

Ineinander einsetzen liefert

$$m\dot{v} = F \qquad \Big| \int_{t_1}^{t_2}$$

$$m v_2 - m v_1 = \int_{t_1}^{t_2} F\, dt$$

$$v'^1 = \frac{m^1 v^1 + m^2 v^2 - e\, m^2 (v^1 - v^2)}{m^1 + m^2}$$

$$v'^2 = \frac{m^1 v^1 + m^2 v^2 + e\, m^1 (v^1 - v^2)}{m^1 + m^2}$$

Diskutiere Extremfälle :

1) voll elast. $e = 1$:

$$v'^1 = \frac{2 m^2 v^2 + (m^1 - m^2) v^1}{m^1 + m^2} \overset{m^1 = m^2 = m}{=} v^2$$

$$v'^2 = \frac{2 m^1 v^1 + (m^2 - m^1) v^2}{m^1 + m^2} = v^1$$

2) voll plastisch $e = 0$:

$$v'^1 = \frac{m^1 v^1 + m^2 v^2}{m^1 + m^2} = v'^2$$