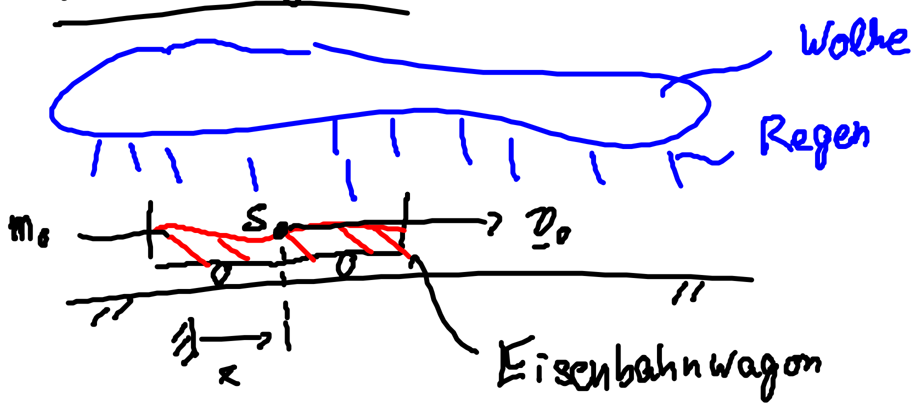
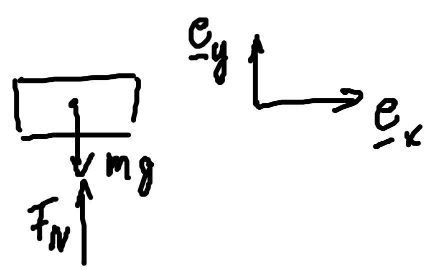
7. Übung 14.06.18

- offenes System
- Drallsatz für eine Punktmasse
- Starrkörperkinematik / Euler Formel
- Momentanpol

Offenes System



Freischnitt:



lex secunda: $\frac{d}{dt}(\underline{p}) = \sum \underline{F}$ mit $\underline{p} = m\underline{v}$

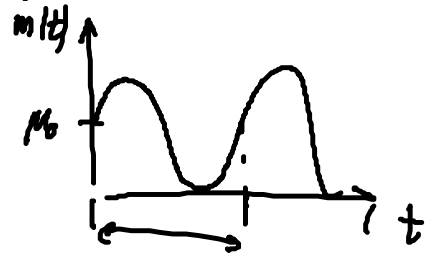
$$\rightarrow \boxed{\frac{d}{dt}(m v_x) = 0} \quad (\underline{e}_x\text{ - Komponente})$$

$$0 = -mg + F_N \quad (\underline{e}_y\text{ - Komponente})$$

$$\dot{m}(t)\, v_x(t) + m(t)\, \dot{v}_x(t) = 0 \qquad (1)$$

Annahme einer zeitabhängigen Massenzufuhr:

$$\dot m(t) = \mu_0 \left(1 + \sin(\omega t)\right)$$



$$\frac{dm}{dt} = \mu_0 \left(1+\sin(\omega t)\right) \quad \Big| \cdot dt \quad (T.d.V.)$$

$$\Rightarrow \int_{m_0}^{m(t)} d\tilde m = \mu_0 \int_0^t \left(1+\sin(\omega \tilde t)\right) d\tilde t$$

$$\Leftrightarrow \tilde m \Big|_{m_0}^{m(t)} = \mu_0 \left[\tilde t - \frac{1}{\omega}\cos(\omega \tilde t)\right]\Big|_0^t$$

$$\Leftrightarrow m(t) - m_0 = \mu_0 \left(t - \frac{1}{\omega}\cos(\omega t) - \left(-\frac{1}{\omega}\right)\right)$$

$$\Rightarrow m(t) = m_0 + \mu_0 \left(t + \frac{1}{\omega}\left(1 - \cos(\omega t)\right)\right)$$

Weiter mit Gl. (1):

$$m(t) \frac{dv_x}{db} = -\dot m(t)\, v_x(t)$$

$$\left[m_0 + \mu_0 \left(t + \frac{1}{\omega}\left(1-\cos(\omega t)\right)\right)\right] \frac{dv_x}{dt} = -\mu_0 \left(1+\sin(\omega t)\right) v_x(t)$$

T.d.V.:

$$\frac{dv_x}{v_x} = -\mu_0 \frac{1+\sin(\omega t)}{m_0 + \mu_0\left(t + \frac{1}{\omega}\left(1-\cos(\omega t)\right)\right)}\, dt$$

$$\int_{v_0}^{v_x(t)} \frac{d\tilde{v}_x}{\tilde{v}_x} = -\mu_0 \underbrace{\int_0^t \frac{1+\sin(\omega\tilde{t})}{m_0+\mu_0\left(\tilde{t}+\frac{1}{\omega}(1-\cos(\omega\tilde{t}))\right)}\, d\tilde{t}}_{=: I_1}$$

Substitution: $z = \mu_0\left(t + \frac{1}{\omega}(1-\cos(\omega t))\right)$

$$\frac{dz}{dt} = \mu_0(1+\sin(\omega t)) \Rightarrow dt = \frac{dz}{\mu_0(1+\sin(\omega t))}$$

$$I_1 = \int \frac{1}{m_0+z}\, dz = \ln(m_0+z) + C$$

$$= \ln\left(m_0 + \mu_0\left(\tilde{t} + \frac{1}{\omega}(1-\cos(\omega\tilde{t}))\right)\right)$$

Einsetzen der Grenzen:

$$I_1 = \ln\left(\frac{m_0+\mu_0\left(t+\frac{1}{\omega}(1-\cos(\omega t))\right)}{m_0}\right)$$

$$\ln\left(\frac{v_x(t)}{v_0}\right) = -\ln\left(\frac{m_0+\mu_0\left(t+\frac{1}{\omega}(1-\cos(\omega t))\right)}{m_0}\right)$$

$$\frac{v_x(t)}{v_0} = \frac{m_0}{m_0+\mu_0\left(t+\frac{1}{\omega}(1-\cos(\omega t))\right)}$$

$$v_x(t) = v_0 \, \frac{m_0}{m_0 + \mu_0 \left(t + \frac{1}{\omega}\left(1-\cos(\omega t)\right)\right)}$$

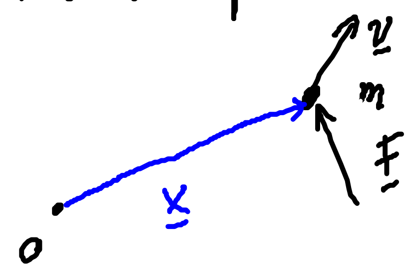
$m_0 = 10^3\,kg$, $V_0 = 40\,\frac{m}{s}$,

$\mu_0 = 3\,kg/s$, $\omega = 10^{-1}\,1/s$

Drallsatz für die Punktmasse

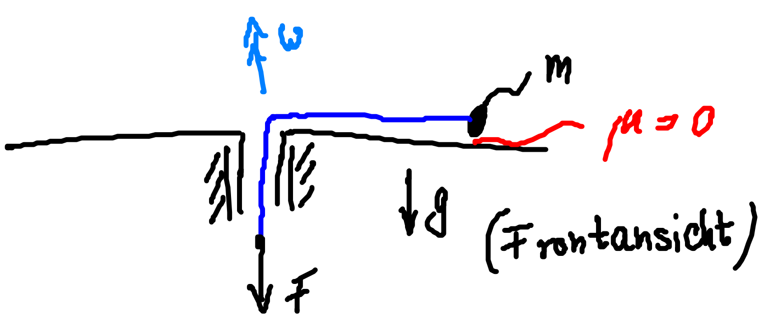
Drall / Drehimpuls von Punktmassen: $\underline{L}^{(0)} = \underline{x} \times \underline{p}$

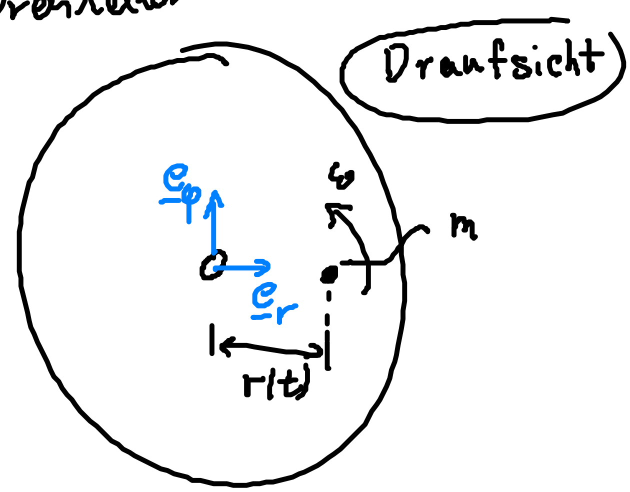
$$= \underline{x} \times m\underline{v}$$



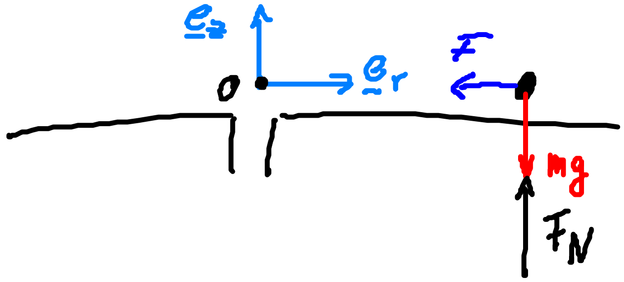
Drallsatz: $\frac{d\underline{L}^{(0)}}{dt} = \underline{x} \times \underline{F} = \underline{M}^{(0)}$

Beispiel: Punktmasse auf einem Drehteller





Freischnitt (Punktmasse)



$$\sum_i \underline{F} = F_N\,\underline{e}_z - mg\,\underline{e}_z - F\,\underline{e}_r$$

2. Newtonsche Gesetz in $\underline{e}_z$-Richtung mit z = konstant

$$m\ddot{z} = F_N - mg \overset{!}{=} 0 \Rightarrow F_N = mg$$

Berechnung von $\underline{M}^{(0)}$:

$$\underline{M}^{(0)} = \sum_i \underline{x}^i \times \underline{F}^i = \underline{x} \times (\cancel{F_N - mg})\,\underline{e}_z + \underline{x} \times (-F\underline{e}_r) \quad \cancel{(F_N - mg)} = 0$$

$$= \underline{x} \times (-F\underline{e}_r) \underset{\underline{x} = r(t)\underline{e}_r}{=} r(t)\underline{e}_r \times (-F\underline{e}_r)$$

$$= -F\,r(t)\,\cancel{\underline{e}_r \times \underline{e}_r} = \underline{0} \quad \cancel{\underline{e}_r \times \underline{e}_r} = \underline{0}$$

$$\frac{d\underline{L}^{(0)}}{dt} = \underline{M}^{(0)} = \underline{0}\,, \qquad \frac{d\underline{L}^{(0)}}{dt} = 0 \Rightarrow \underline{L}^{(0)} = \text{konst.}$$

$$\underline{L}^{(0)} = \underline{x} \times m\underline{v} = r(t)\underline{e}_r \times m\left(\dot{r}(t)\underline{e}_r + r(t)\dot{\varphi}(t)\underline{e}_\varphi\right)$$

$$= m\,r(t)\dot{r}(t)\,\cancel{\underline{e}_r \times \underline{e}_r} + m\,r^2(t)\dot{\varphi}(t)\,\underline{e}_r \times \underline{e}_\varphi \quad \cancel{\underline{e}_r \times \underline{e}_r} = \underline{0}, \quad \underline{e}_r \times \underline{e}_\varphi = \underline{e}_z$$

$$= m\,r^2(t)\,\dot{\varphi}(t)\,\underline{e}_z \underset{\dot{\varphi} = \omega}{=} m\,r^2(t)\,\omega(t)\,\underline{e}_z$$

Wie ändert sich $\omega = \dot{\varphi}$ falls der Radius halbiert wird?

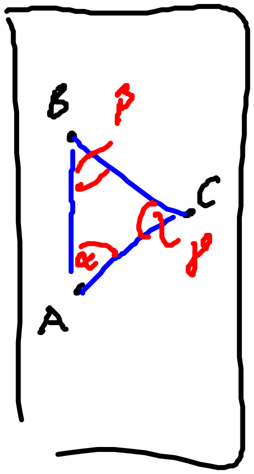
$$\underline{L}_1^{(0)} = \underline{L}_2^{(0)}\,, \quad \underline{L}_1^{(0)} = m R^2 \omega_1 \underline{e}_z\,, \quad \underline{L}_2^{(0)} = m\left(\frac{R}{2}\right)^2 \omega_2 \underline{e}_z$$

$$\Rightarrow m R^2 \omega_1 = m\frac{R^2}{4}\omega_2 \Leftrightarrow \underline{\underline{\omega_2 = 4\,\omega_1}}$$

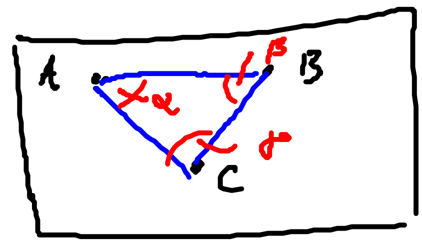
<u>Starrkörperkinematik / Euler-Formel</u>

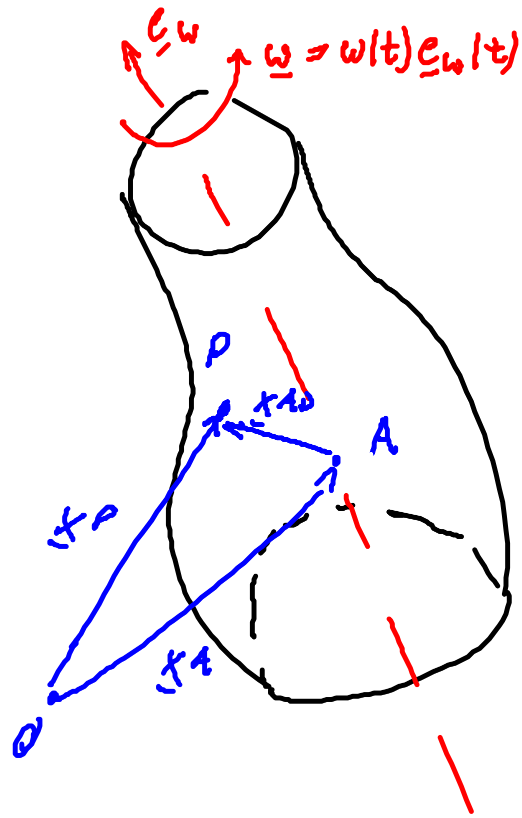
Starrer Körper: Keine Deformation möglich! D. h. Abstand zwischen zwei beliebigen Punkten bleibt konstant. Relative Lagen von drei Punkten ändern sich nicht.

$\underline{t = t_1}$



$\underline{t = t_2}$





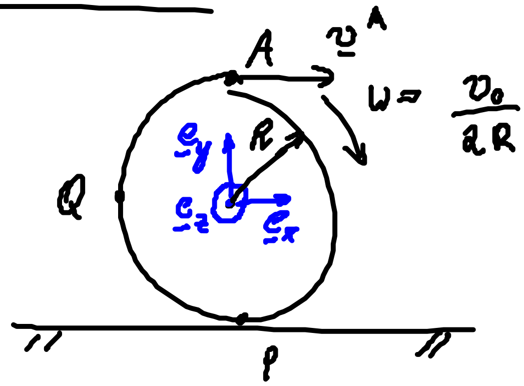
Eulersche Kinematikgleichungen

$$\underline{x}^P(t) = \underline{x}^A(t) + \underline{x}^{AP}(t)$$

$$\underline{v}^P(t) = \underline{v}^A(t) + \underline{\omega}(t) \times \underline{x}^{AP}(t)$$

$$\underline{a}^P(t) = \underline{a}^A(t) + \dot{\underline{\omega}}(t) \times \underline{x}^{AP}(t) + \underline{\omega}(t) \times \left(\underline{\omega}(t) \times \underline{x}^{AP}(t)\right)$$

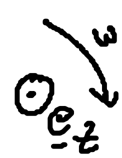
1. Beispiel



$\underline{v}^A = 3 v_0 \underline{e}_x$

gesucht: $\underline{v}^P$ und $\underline{v}^Q$

gegeben: $\underline{v}^A$, $\underline{\omega} = -\frac{v_0}{2R} \underline{e}_z$



$$\underline{v}^P = \underline{v}^A + \underline{\omega} \times \underline{x}^{AP}$$

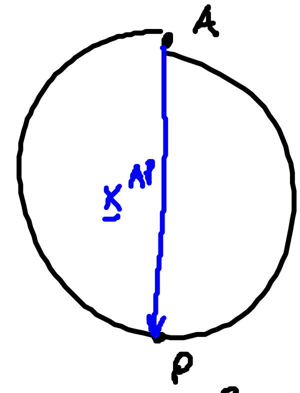
$$= 3 v_0 \underline{e}_x + (-\omega \underline{e}_z) \times \underline{x}^{AP}, \qquad \underline{x}^{AP} = \underline{x}^P - \underline{x}^A = -2R\underline{e}_y$$

$$= 3 v_0 \underline{e}_x + \left(-\frac{v_0}{2R} \underline{e}_z\right) \times (-2R \underline{e}_y)$$

$$= 3 v_0 \underline{e}_x + v_0 \, \underline{e}_z \times \underline{e}_y$$

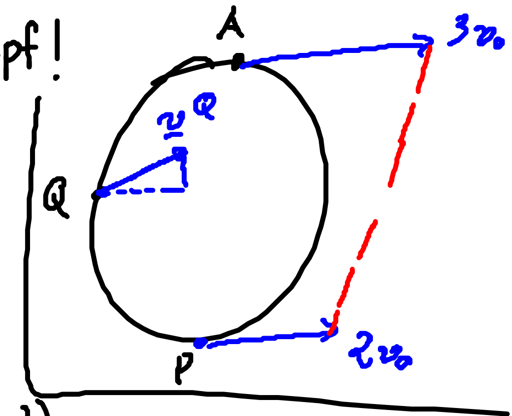
$$= (3 v_0 - v_0) \underline{e}_x$$

$$= 2 v_0 \underline{e}_x$$



kein reines Rollen, da $\underline{v}^P \neq 0$, es gibt Schlupf!

$\underline{x}^A = R\underline{e}_y$, $\underline{x}^P = -R\underline{e}_y$



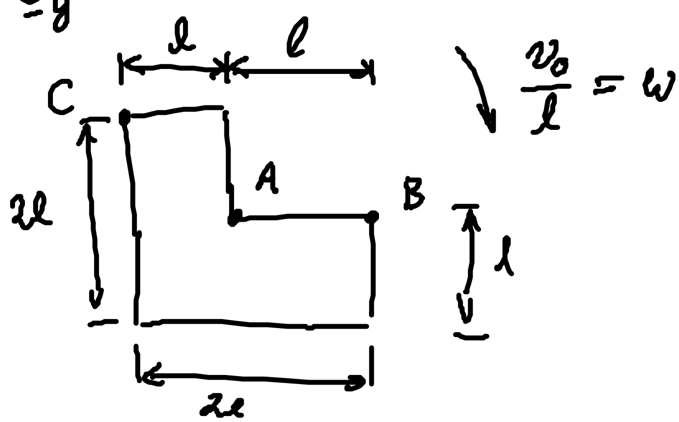
$$\underline{v}^Q = \underline{v}^A + \underline{\omega} \times \underline{x}^{AQ}, \qquad \underline{x}^{AQ} = -R(\underline{e}_x + \underline{e}_y)$$

$$= 3 v_0 \underline{e}_x + \left(-\frac{v_0}{2R} \underline{e}_z\right) \times \left(-R(\underline{e}_x + \underline{e}_y)\right)$$

$$= 3 v_0 \underline{e}_x + \frac{v_0}{2} \left(\overbrace{\underline{e}_z \times \underline{e}_x}^{\underline{e}_y} + \overbrace{\underline{e}_z \times \underline{e}_y}^{-\underline{e}_x} \right)$$

$$= \frac{5}{2} v_0 \underline{e}_x + \frac{v_0}{2} \underline{e}_y$$

2. Beispiel



$$\underline{v}^A = \frac{\sqrt{3}}{2} v_0 \underline{e}_x + \frac{1}{2} v_0 \underline{e}_y$$

gesucht: $\underline{v}^C$, $\underline{v}^B$ → Hausaufgabe

Momentanpol