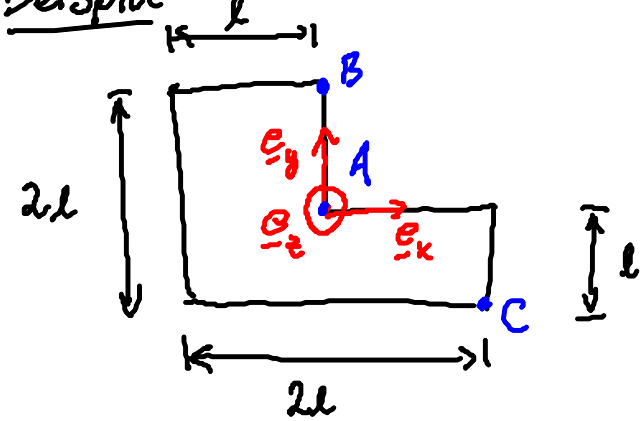
8. große Übung

- Wiederholung: Eulersche Kinematikgleichung
- Wiederholung: Momentanpol
- Massenträgheitstensor
- Starrkörperkinetik in der Ebene

Eulersche Kinematikgleichungen

$$\underline{x}^B = \underline{x}^A + \underline{x}^{AB}, \quad \underline{v}^B = \underline{v}^A + \underline{\omega} \times \underline{x}^{AB}, \quad \underline{a}^B = \underline{a}^A + \dot{\underline{\omega}} \times \underline{x}^{AB} + \underline{\omega} \times (\underline{\omega} \times \underline{x}^{AB})$$

Beispiel



geg.: $\underline{v}^B = \left(\frac{\sqrt{3}}{2} - 1\right) v_0 \underline{e}_x + \frac{v_0}{2} \underline{e}_y$

$\underline{v}^C = \left(\frac{\sqrt{3}}{2} + 1\right) v_0 \underline{e}_x + \frac{3}{2} v_0 \underline{e}_y$

gesucht: $\underline{\omega} = \omega \underline{e}_z$, Π: Lage des Momentanpols, $\underline{v}^A$

→ Bestimmung von $\underline{\omega}$:

$$\underline{v}^A = \underline{v}^B + \underline{\omega} \times \underline{x}^{BA}, \quad \underline{v}^A = \underline{v}^C + \underline{\omega} \times \underline{x}^{CA}$$

$$\Longrightarrow \underline{v}^A - \underline{v}^A = \boxed{\underline{0} = \underline{v}^B + \underline{\omega} \times \underline{x}^{BA} - \underline{v}^C - \underline{\omega} \times \underline{x}^{CA}}$$

Unbekannte: $\underline{\omega}$

$$\underline{v}^B - \underline{v}^C = \underline{\omega} \times \underline{x}^{CA} - \underline{\omega} \times \underline{x}^{BA} = \underline{\omega} \times (\underline{x}^{CA} - \underline{x}^{BA})$$

mit $\underline{x}^{CA} - \underline{x}^{BA} = \underbrace{(\underline{x}^{A} - \underline{x}^{C})}_{\underline{x}^{CA}} - \underbrace{(\underline{x}^{A} - \underline{x}^{B})}_{\underline{x}^{BA}} = \underline{x}^{B} - \underline{x}^{C} = \underline{x}^{CB}$

$$\boxed{\underline{v}^{B} - \underline{v}^{C} = \underline{\omega} \times \underline{x}^{CB}} \rightarrow \underline{v}^{B} = \underline{v}^{C} + \underline{\omega} \times \underline{x}^{CB} \quad \text{(Euler Formel)}$$

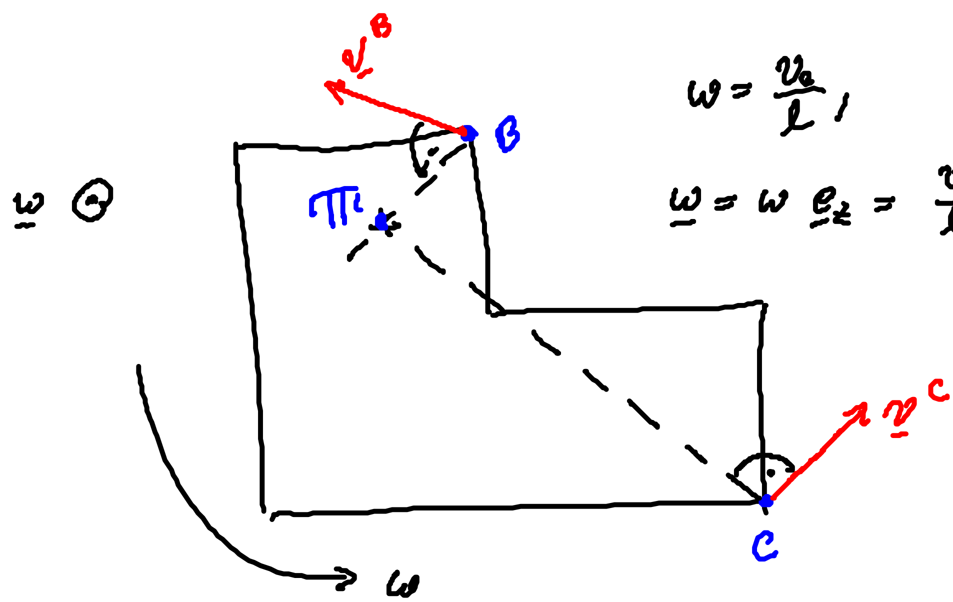
$$\underline{x}^{B} = l\,\underline{e}_y \,, \quad \underline{x}^{C} = l\,\underline{e}_x - l\,\underline{e}_y \,, \quad \underline{x}^{CB} = -l\,\underline{e}_x + 2l\,\underline{e}_y$$

$$\underline{v}^{B} - \underline{v}^{C} = -2v_0\,\underline{e}_x - v_0\,\underline{e}_y = \omega\,\underline{e}_z \times (-l\,\underline{e}_x + 2l\,\underline{e}_y)$$

$$= -\omega l\,\underline{e}_y - 2\omega l\,\underline{e}_x$$

- $\underline{e}_x$-Komponente: $-2v_0 = -2\omega l \Rightarrow \omega = v_0/l$
- $\underline{e}_y$-Komponente: $-v_0 = -\omega l \Rightarrow \omega = v_0/l$

} beide Komponenten liefern dasselbe Ergebnis!
→ Möglichkeit zur Überprüfung



$\omega = \frac{v_0}{l}$,

$\underline{\omega} = \omega\,\underline{e}_z = \frac{v_0}{l}\,\underline{e}_z$

→ Bestimmung des Momentanpols Π:

bekannt: $\underline{v}^{B}$, $\underline{\omega}$, $\underline{v}^{C}$

$$\underline{v}^{\Pi} = \underline{v}^{A} + \underline{\omega} \times \underline{x}^{A\Pi} = \underline{v}^{B} + \underline{\omega} \times \underline{x}^{B\Pi}$$

$$\underline{v}^{\Pi} \overset{!}{=} \underline{0} \Leftrightarrow \underline{v}^{B} + \underline{\omega} \times \underline{x}^{B\Pi} = \underline{0} \,, \quad -\underline{x}^{B\Pi} = \underline{x}^{\Pi B}$$

$$\Leftrightarrow \quad \underline{v}^B = \underline{\omega} \times \underline{x}^{TTB} = \underline{\omega} \times (\underline{x}^B - \underline{x}^{TT})$$

$$\underline{x}^{TTB} = \underline{x}^B - \underline{x}^{TT}$$

$$\underline{\omega} \times \underline{x}^{TT} = -\underline{v}^B + \underline{\omega} \times \underline{x}^B \quad \Big| \; \underline{\omega} \times$$

$$\underline{\omega} \times (\underline{\omega} \times \underline{x}^{TT}) = -\underline{\omega} \times \underline{v}^B + \underline{\omega} \times (\underline{\omega} \times \underline{x}^B) \quad (1)$$

$$\underline{\omega} \times (\underline{\omega} \times \underline{x}^{TT}) = \underbrace{(\underline{\omega} \cdot \underline{x}^{TT})}_{\to 0} \underline{\omega} + (\underline{\omega} \cdot \underline{\omega}) \underline{x}^{TT} = \omega^2 \underline{x}^{TT} \quad (2)$$

$$\underline{a} \times \underline{b} \times \underline{c} = (\underline{a} \cdot \underline{c}) \, \underline{b} + (\underline{b} \cdot \underline{a}) \, \underline{c}$$

$$\underline{\omega} \times (\underline{\omega} \times \underline{x}^B) = \underbrace{(\underline{\omega} \cdot \underline{x}^B)}_{\to 0} \underline{\omega} + (\underline{\omega} \cdot \underline{\omega}) \underline{x}^B = \omega^2 \underline{x}^B \quad (3)$$

(2), (3) in (1):

$$\omega^2 \underline{x}^{TT} = -\underline{\omega} \times \underline{v}^B + \omega^2 \underline{x}^B$$

$$\Rightarrow \underline{x}^{TT} = -\frac{1}{\omega^2} \underline{\omega} \times \underline{v}^B + \underline{x}^B$$

$$= -\frac{1}{\left(\frac{v_0}{l}\right)^2} \frac{v_0}{l} \underline{e}_z \times \left(\left(\frac{\sqrt{3}}{2} - 1\right) v_0 \underline{e}_x + \frac{v_0}{2} \underline{e}_y \right) + l \underline{e}_y$$

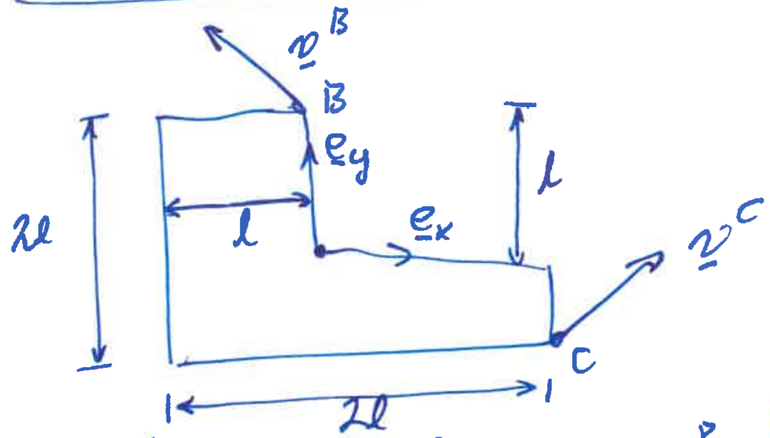
$$= -l \left[\left(\frac{\sqrt{3}}{2} - 1\right) \underline{e}_y - \frac{1}{2} \underline{e}_x \right] + l \underline{e}_y$$

$$= l \left(\frac{1}{2} \underline{e}_x - \frac{\sqrt{3}}{2} \underline{e}_y \right)$$

Falsch!

Vorzeichenfehler bei der Graßmann-Identität (bac-cab-Formel)

Bestimmung der Lage des Momentanpols Π



bekannt: $\underline{\omega} = \frac{v_0}{l}\, \underline{e}_z\,,\quad \underline{v}^B = \left(\frac{\sqrt{3}}{2} - 1\right) v_0\, \underline{e}_x + \frac{v_0}{2}\, \underline{e}_y\,.$

gesucht: $\underline{x}^{\Pi}$.

Eulersche Kinematikgleichung:

$$\underline{v}^{\Pi} = \underline{v}^B + \underline{\omega} \times \underline{x}^{B\Pi}\,. \quad (1)$$

Bedingung des Momentanpols: $\underline{v}^{\Pi} = \underline{0}$ (momentane Ruhe)

Gl. (1) mit $\underline{v}^{\Pi} = \underline{0}$: $\quad \underline{v}^B = -\underline{\omega} \times \underline{x}^{B\Pi} = \underline{\omega} \times (-\underline{x}^{B\Pi}) = \underline{\omega} \times \underline{x}^{\Pi B}$. (2)

mit $\underline{x}^{\Pi B} = \underline{x}^B - \underline{x}^{\Pi}$ ergibt sich Gl. (2) zu:

$$\underline{v}^B = \underline{\omega} \times \underline{x}^B - \underline{\omega} \times \underline{x}^{\Pi} \Leftrightarrow \underline{\omega} \times \underline{x}^{\Pi} = -\underline{v}^B + \underline{\omega} \times \underline{x}^B\,. \quad (3)$$

Gl. (3) enthält die Unbekannte $\underline{x}^{\Pi}$. Der Vektor $\underline{x}^B$ ist aus der Geometrie bekannt:

$$\underline{x}^B = l\, \underline{e}_y\,.$$

Mit der Graßmann-Identität (bac – cab Formel)

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\, \underline{b} - (\underline{a} \cdot \underline{b})\, \underline{c}$$

ergibt $\underline{\omega} \times$ Gl. (3):

$$\underline{\omega} \times (\underline{\omega} \times \underline{x}^{\Pi}) = -\underline{\omega} \times \underline{v}^B + \underline{\omega} \times (\underline{\omega} \times \underline{x}^B) \quad (4)$$

folgende Terme:

$$\underline{\omega} \times (\underline{\omega} \times \underline{x}^{\Pi}) = \underbrace{(\underline{\omega} \cdot \underline{x}^{\Pi})}_{\to 0} \underline{\omega} - (\underline{\omega} \cdot \underline{\omega}) \underline{x}^{\Pi} = -\underline{\omega}^2 \underline{x}^{\Pi},$$

$$\underline{\omega} \times (\underline{\omega} \times \underline{x}^{B}) = \underbrace{(\underline{\omega} \cdot \underline{x}^{B})}_{\to 0} \underline{\omega} - (\underline{\omega} \cdot \underline{\omega}) \underline{x}^{B} = -\underline{\omega}^2 \underline{x}^{B}.$$

Die Skalarprodukte verschwinden, da $\underline{\omega} \parallel \underline{e}_z$ ist und $\underline{x}^{\Pi} \perp \underline{e}_z$ und $\underline{x}^{B} \perp \underline{e}_z$ sind.
($\parallel$: parallel , $\perp$: senkrecht)

Einsetzen in Gl. (4):

$$-\underline{\omega}^2 \underline{x}^{\Pi} = -\underline{\omega} \times \underline{v}^B - \underline{\omega}^2 \underline{x}^B$$

$$\Leftrightarrow \quad \underline{x}^{\Pi} = \frac{1}{\omega^2} \underline{\omega} \times \underline{v}^B + \underline{x}^B$$

$$= \frac{l^2}{v_0^2} \frac{v_0}{l} \underline{e}_z \times \left[\left(\frac{\sqrt{3}}{2} - 1\right) v_0 \underline{e}_x + \frac{v_0}{2} \underline{e}_y \right] + l \, \underline{e}_y$$

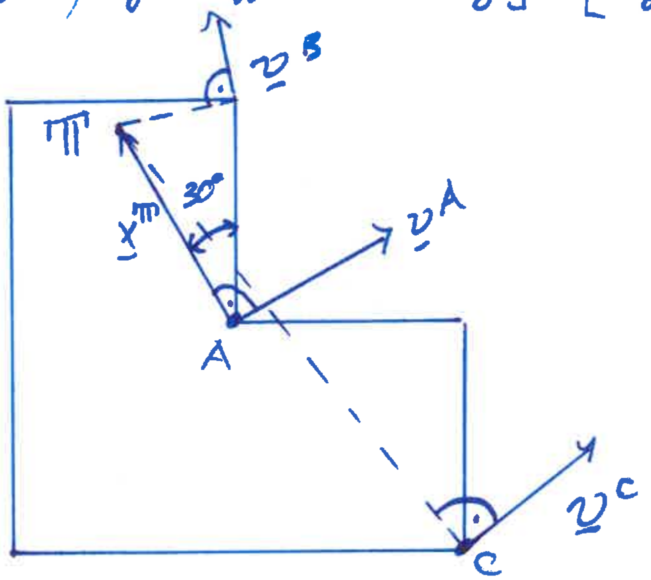
$$= l \left[\underline{e}_z \times \left[\left(\frac{\sqrt{3}}{2} - 1\right) \underline{e}_x + \frac{1}{2} \underline{e}_y \right] + \underline{e}_y \right]$$

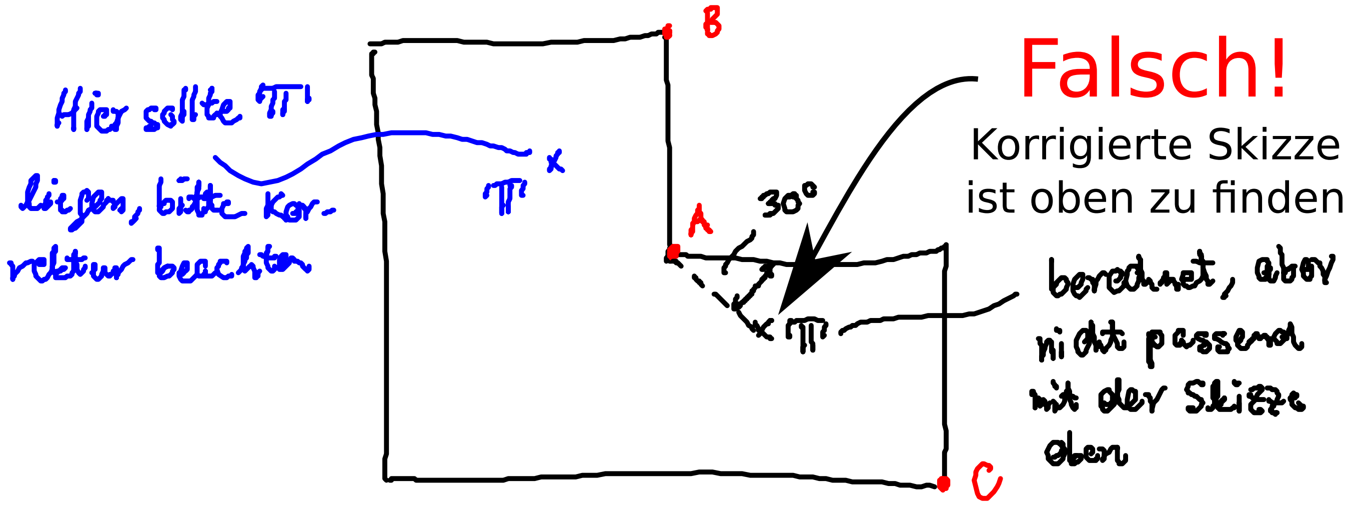
$$= l \left[\left(\frac{\sqrt{3}}{2} - 1\right) \underline{e}_y - \frac{1}{2} \underline{e}_x + \underline{e}_y \right] = \left[-\frac{1}{2} \underline{e}_x + \frac{\sqrt{3}}{2} \underline{e}_y \right] l$$

Graphische Darstellung:

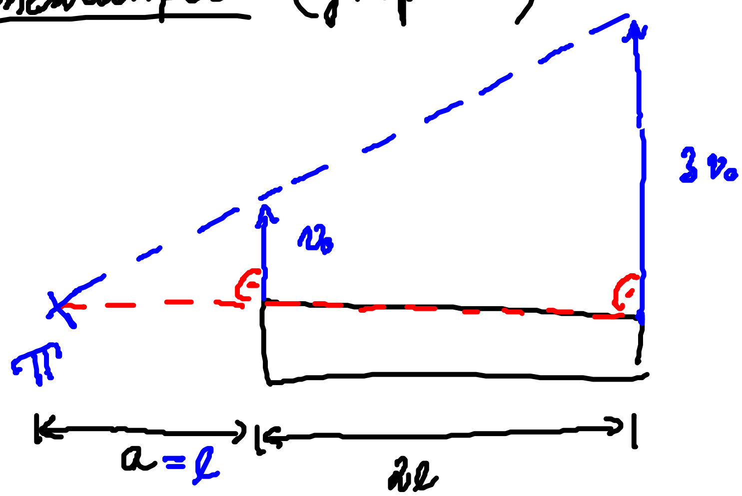
$l \mathrel{\hat{=}} 3$ cm

Momentanpol ist maßstabsgetreu eingezeichnet. Geschwindigkeiten sind nicht maßstabsgetreu.





Momentanpol (graphisch)



$$\begin{cases} v_0 = \omega\, a \\ 3v_0 = \omega\,(a + 2\ell) \end{cases}$$

$$\begin{cases} \omega = \dfrac{v_0}{\ell} \\ a = \ell \end{cases}$$

Massenträgheitstensor

$\underline{\underline{\Theta}}^{(A)}$: beschreibt den Widerstand eines Körpers gegenüber der Änderung der Drehbewegung bzgl. Punktes A.

$$\frac{d}{dt}\left(\underline{L}^{(A)}\right) = \underline{M}^{(A)}, \quad \underline{L}^{(A)} = \underline{\underline{\Theta}}^{(A)} \cdot \underline{\omega}$$

Analogie zur Translation (Newton)

$$\frac{d}{dt}\,\underline{p} = \underline{F}, \quad \underline{p} = m\,\underline{v}$$

→ Translation: nur die Gesamtmasse geht ein

→ Rotation: Verteilung ist relevant.

$$\underline{\underline{\Theta}}^{(A)} = \int \left(\underline{x}^{AP} \cdot \underline{x}^{AP} \, \underline{\underline{I}} - \underline{x}^{AP} \underline{x}^{AP} \right) dm$$

$$\Theta^{(A)} = \begin{bmatrix} \Theta_{xx}^{(A)} & \Theta_{xy}^{(A)} & \Theta_{xz}^{(A)} \\ \Theta_{yx}^{(A)} & \Theta_{yy}^{(A)} & \Theta_{yz}^{(A)} \\ \Theta_{zx}^{(A)} & \Theta_{zy}^{(A)} & \Theta_{zz}^{(A)} \end{bmatrix}$$

$$\Theta_{xx}^{(A)} = \int \left[(x-x^{(A)})^2 + (y-y^{(A)})^2 + (z-z^{(A)})^2 - (x-x^{(A)})(x-x^{(A)}) \right] dm$$

$$= \int \left[(y-y^{(A)})^2 + (z-z^{(A)})^2 \right] dm$$

$$\Theta_{yy}^{(A)} = \int \left[(x-x^{(A)})^2 + (z-z^{(A)})^2 \right] dm$$

$$\Theta_{zz}^{(A)} = \int \left[(x-x^{(A)})^2 + (y-y^{(A)})^2 \right] dm$$

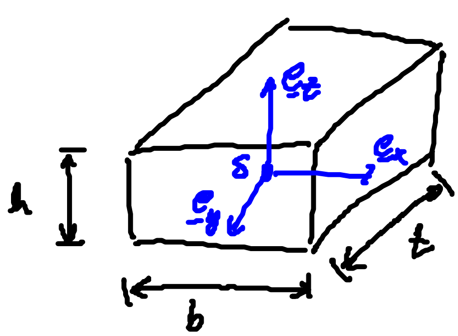
$$\Theta_{xy}^{(A)} = - \int (x-x^{(A)})(y-y^{(A)}) \, dm = \Theta_{yx}^{(A)}$$

$$\Theta_{xz}^{(A)} = - \int (x-x^{(A)})(z-z^{(A)}) \, dm = \Theta_{zx}^{(A)}$$

$$\Theta_{yz}^{(A)} = - \int (y-y^{(A)})(z-z^{(A)}) \, dm = \Theta_{zy}^{(A)}$$

1. Beispiel – Quader

gesucht: $\underline{\underline{\Theta}}^{(S)}$



$$\Theta_{xx}^{(S)} = \int \left[(y-y^{(S)})^2 + (z-z^{(S)})^2 \right] dm$$

$$x^{(S)} = y^{(S)} = z^{(S)} = 0, \quad dm = \rho_0\, dV = \rho_0\, dx\, dy\, dz$$

$$\Theta_{xx}^{(S)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{t}{2}}^{\frac{t}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} (y^2 + z^2)\, \rho_0\, dx\, dy\, dz$$

$$= \rho_0\, b \int_{-\frac{h}{2}}^{h/2} \int_{-\frac{t}{2}}^{t/2} (y^2 + z^2)\, dy\, dz$$

$$= \rho_0\, b \int_{-h/2}^{h/2} \left(\frac{1}{3} y^3 + z^2 y\right)\Big|_{-t/2}^{t/2} dz$$

$$= \rho_0\, b \int_{-h/2}^{h/2} \left(\frac{1}{12} t^3 + z^2 t\right) dz = \rho_0\, b t \left(\frac{1}{12} t^2 z + \frac{1}{3} z^3\right)\Big|_{-h/2}^{h/2}$$

$$= \rho_0\, b t \left(\frac{1}{12} t^2 h + \frac{1}{12} h^3\right) = \frac{\rho_0\, b h t}{12} (t^2 + h^2)$$

$$= \frac{m}{12} (h^2 + t^2)$$

$m = \rho_0 V = \rho_0 b h t$

h: Ausdehnung in z-Richtung

t: Ausdehnung in y-Richtung

$$\Theta_{yy}^{(S)} = \rho_0 \int_{-\frac{h}{2}}^{h/2} \int_{-\frac{t}{2}}^{\frac{t}{2}} \int_{-\frac{b}{2}}^{b/2} [x^2 + z^2]\, dx\, dy\, dz$$

$$= \frac{m}{12} (b^2 + h^2), \quad \Theta_{zz}^{(S)} = \frac{m}{12} (b^2 + t^2)$$

In $\Theta_{yy}^{(S)}$: b: x-Richtung, h: z-Richtung

In $\Theta_{zz}^{(S)}$: b: x, t: y

$$\Theta_{xy}^{(S)} = - \int x\, y\, dm$$

$$= -\rho_0 h \int_{-\frac{t}{2}}^{t/2} \int_{-\frac{b}{2}}^{b/2} x\, y\, dx\, dy = -\rho_0 h \int_{-t/2}^{t/2} \frac{x^2}{2}\Big|_{-b/2}^{b/2} y\, dy$$

$$= 0 \text{ wegen Symmetric!}$$

$$\Theta_{xy}^{(S)} = \Theta_{xz}^{(S)} = \Theta_{yz}^{(S)} = 0$$