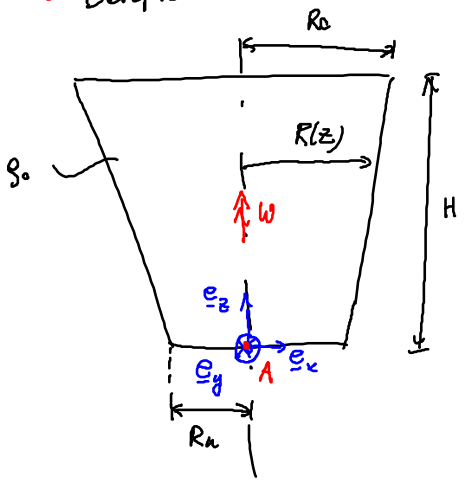
9. große Übung

- Massenträgheitsmoment d. Kegelstumpfes (Döner)
- Massenträgheitsmomente und der Satz von Steiner
- Starrkörperkinetik in der Ebene
- Beispiel: Hula hoop Reifen



gesucht: $\underline{\underline{\Theta}}^{(A)}$

Zylinderkoordinaten: r, φ, z

$R = R(z) = R_u + \frac{R_o - R_u}{H} z$, (Außenradius)

$x = r\cos(\varphi)$, $\quad dV = r\, dr\, d\varphi\, dz$

$y = r\sin(\varphi)$, $\quad dm = \varrho_o\, dV$

$z = z$,

$$\Theta_{zz}^{(A)} = \int (x^2 + y^2)\, dm = \int_0^H \int_0^{2\pi} \int_0^{R(z)} \left((r\cos(\varphi))^2 + (r\sin(\varphi))^2 \right) \varrho_o\, r\, dr\, d\varphi\, dz$$

$$= \varrho_o \int_0^H \int_0^{2\pi} \int_0^{R(z)} r^3\, dr\, d\varphi\, dz = \cancel{2}\pi \varrho_o \int_0^H \frac{r^4}{\cancel{2}4}\Big|_0^{R(z)} dz$$

$$\boxed{\int (\alpha + \beta x)^n dx = \frac{1}{(n+1)\beta} (\alpha + \beta x)^{n+1} + C}$$

$$= \frac{\pi}{2} \varrho_o \int_0^H \left(R_u + \frac{R_o - R_u}{H} z \right)^4 dz$$

$$= \frac{\pi}{2} \varrho_o \frac{1}{5} \frac{\left(R_u + \frac{R_o - R_u}{H} z \right)^5}{\left(\frac{R_o - R_u}{H} \right)} \Bigg|_0^H = \frac{\varrho_o \pi}{10} \frac{H}{R_o - R_u} \left[\left(R_u + \frac{R_o - R_u}{\cancel{H}} \cancel{H} \right)^5 - R_u^5 \right]$$

$$= \frac{\pi}{10} \varrho_o \frac{H}{R_o - R_u} \left(R_o^5 - R_u^5 \right)$$

mit $m = \varrho_o V$, $V = \frac{\pi}{3} H \left(R_o^2 + R_u R_o + R_u^2 \right)$, $\varrho_o = \frac{m}{V} = \frac{3m}{\pi H} \frac{1}{R_o^2 + R_u R_o + R_u^2}$

Substitution von ϱ_o durch m/V

$$\Theta_{zz}^{(A)} = \frac{\cancel{\pi}}{10} \frac{3m}{\cancel{\pi} \cancel{H}} \frac{\cancel{H}}{R_o^2 + R_u R_o + R_u^2} \frac{R_o^5 - R_u^5}{(R_o - R_u)}, \text{ mit } (R_o - R_u)(R_o^2 + R_u R_o + R_u^2)$$

$$= R_o^3 + \cancel{R_u R_o^2} + \cancel{R_o R_u^2} - \cancel{R_u R_o^2} - \cancel{R_u^2 R_o} - R_u^3$$

$$= R_o^3 - R_u^3$$

$$= \frac{3}{10} m \frac{R_o^5 - R_u^5}{R_o^3 - R_u^3}$$ ← Formel in Formelsammlung (Dubbel)

Deviationsmomente:

$$\Theta_{yz}^{(A)} = -\int y\,z\,dm = -\rho_0 \int_0^H \int_0^{2\pi} \int_0^{R(z)} r\sin(\varphi)\, z \; r\,dr\,d\varphi\,dz$$

$$= -\rho_0 \int_0^H \int_0^{R(z)} z\,r^2\,dr\,dz \int_0^{2\pi} \sin(\varphi)\,d\varphi = -\rho_0 \int_0^H \int_0^{R(z)} z\,r^2\,dr\,dz \left(\cancel{-\cos(\varphi)\Big|_0^{2\pi}}^{\,0}\right) = 0$$

$\rightarrow$ Symmetrieachse

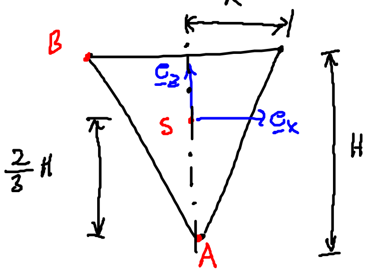
$$\Theta_{xz}^{(A)} = -\int x\,z\,dm = -\rho_0 \int_0^H \int_0^{2\pi} \int_0^{R(z)} r\cos(\varphi)\, z \; r\,dr\,d\varphi\,dz = 0$$

Drall für $\underline{\omega} = \omega\, \underline{e}_z$

$$\underline{L}^{(A)} = \underline{\underline{\Theta}}^{(A)} \cdot \underline{\omega} = \begin{bmatrix} \Theta_{xx} & \Theta_{xy} & \Theta_{xz} \\ & \Theta_{yy} & \Theta_{yz} \\ \text{symm.} & & \Theta_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = \begin{pmatrix} \Theta_{xz}\,\omega \\ \Theta_{yz}\,\omega \\ \Theta_{zz}\,\omega \end{pmatrix} = \cancel{\Theta_{xz}^{(A)}}^{\,0}\,\omega\,\underline{e}_x + \cancel{\Theta_{yz}^{(A)}}^{\,0}\,\omega\,\underline{e}_y + \Theta_{zz}^{(A)}\,\omega\,\underline{e}_z$$

$$= \Theta_{zz}^{(A)}\,\omega\,\underline{e}_z$$

Massenträgheitstensor und der Satz von Steiner



Kegel: Spezialfall des Kegelstumpfes mit $R_u = 0$, $R_o = R$

$$\Theta_{zz}^{(A)} = \frac{3}{10}\, m\, \frac{R_o^5 - R_u^5}{R_o^3 - R_u^3} = \frac{3}{10}\, m\, R^2$$

$R_o = R$
$R_u = 0$

Was ist mit dem Deviationsmoment $\Theta_{xy}^{(A)}$?

$$\Theta_{xy}^{(A)} = -\int x\,y\,dm = -\rho_0 \int_0^H \int_0^{2\pi} \int_0^{R(z)} r\cos(\varphi)\; r\sin(\varphi)\; r\,dr\,d\varphi\,dz = -\rho_0 \int_0^H \int_0^{R(z)} r^3\,dr\,dz \int_0^{2\pi} \sin(\varphi)\cos(\varphi)\,d\varphi$$

$$= -\rho_0 \int_0^H \int_0^{R(z)} r^3\,dr\,dz \; \cancel{\frac{1}{2}\sin^2(\varphi)\Big|_0^{2\pi}}^{\,0} = 0$$

Formelsammlung:

$$\int \sin(x)\cos(x)\,dx = \frac{1}{2}\sin^2(x)$$

Substitution:

$$\eta = \sin(\varphi) \qquad \frac{d\eta}{d\varphi} = \cos(\varphi) \Rightarrow d\varphi = \frac{d\eta}{\cos(\varphi)}$$

$$\int \sin(\varphi)\cos(\varphi)\,d\varphi = \int \eta\,d\eta = \frac{1}{2}\eta^2 = \frac{1}{2}\sin^2(\varphi)$$

keine Deviationsmomente bei Rotation um A.

- Wie berechnen $\underline{\underline{\Theta}}$ bzgl. des Schwerpunktes S und des Punktes B?

$\rightarrow$ Satz von Steiner für Massenträgheitsmomente:

$$\underline{\underline{\Theta}}^{(B)} = \underline{\underline{\Theta}}^{(S)} + \underbrace{m\left(\underline{x}^{SB} \cdot \underline{x}^{SB}\; \underline{\underline{I}} - \underline{x}^{SB} \otimes \underline{x}^{SB}\right)}_{\text{Steiner Anteil}}$$

$$\underline{x}^{SB} = \underline{x}^{B} - \underline{x}^{S}$$

$$\Theta_{xx}^{(B)} = \Theta_{xx}^{(S)} + m\left((x^B - x^S)^2 + (y^B - y^S)^2 + (z^B - z^S)^2 - (x^B - x^S)(x^B - x^S)\right) = \Theta_{xx}^{(S)} + m\left((y^B - y^S)^2 + (z^B - z^S)^2\right)$$

$$\Theta_{yy}^{(B)} = \Theta_{yy}^{(S)} + m\left((x^B - x^S)^2 + (z^B - z^S)^2\right)$$

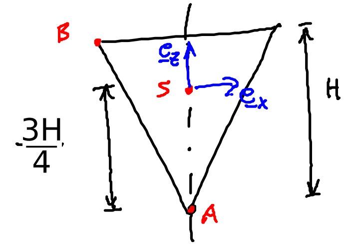
R

$$\Theta_{zz}^{(B)} = \Theta_{zz}^{(S)} + m\left((x^B - x^S)^2 + (y^B - y^S)^2\right)$$

$$\Theta_{xy}^{(B)} = \Theta_{xy}^{(S)} - m(x^B - x^S)(y^B - y^S)$$

$$\Theta_{xz}^{(B)} = \Theta_{xz}^{(S)} - m(x^B - x^S)(z^B - z^S)$$

$$\Theta_{yz}^{(B)} = \Theta_{yz}^{(S)} - m(y^B - y^S)(z^B - z^S)$$



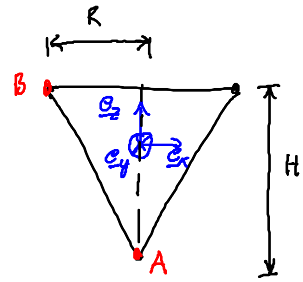
$\rightarrow$ $\underline{\Theta}^{(S)}$ für den Kegel (bzgl. des Schwerpunktes)

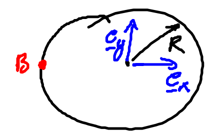
$$\Theta_{zz}^{(S)} = \Theta_{zz}^{(A)} + m\left((\overset{0}{\cancel{x^S - x^A}})^2 + (\overset{0}{\cancel{y^S - y^A}})^2\right) \quad \text{mit } \underline{x}^{AS} = \frac{3H}{4}\underline{e}_z,\ x^S - x^A = 0,\ y^S - y^A = 0$$

$$= \Theta_{zz}^{(A)}$$

$$\Theta_{xx}^{(S)} = \Theta_{xx}^{(A)} + m\left((\overset{0}{\cancel{y^S - y^A}})^2 + (\overset{\frac{2}{3}H}{z^S - z^A})^2\right) = \Theta_{xx}^{(A)} + m\,\frac{9}{16}H^2$$

$$\Theta_{xy}^{(S)} = \Theta_{xy}^{(A)} - m\,(\overset{0}{\cancel{x^S - x^A}})(\overset{0}{\cancel{y^S - y^A}}) = \Theta_{xy}^{(A)} = 0$$





$\rightarrow$ $\underline{\Theta}^{(B)}$ für den Kegel (bzgl. des "Eckpunktes")

$$\underline{x}^{SB} = -R\,\underline{e}_x + \frac{H}{4}\underline{e}_z$$

$$z^B - z^S = \frac{H}{4}, \quad x^B - x^S = -R$$

$$\Theta_{xx}^{(B)} = \Theta_{xx}^{(A)} + m\left((\overset{0}{\cancel{y^B - y^S}})^2 + (z^B - z^S)^2\right) = \Theta_{xx}^{(A)} + m\,\frac{H^2}{16}$$

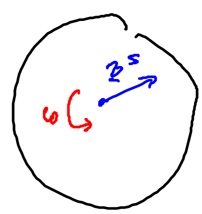
$$\Theta_{yy}^{(B)} = \dots, \quad \Theta_{zz}^{(B)} = \dots$$

$$\Theta_{xz}^{(B)} = \Theta_{xz}^{(A)} - m(x^B - x^S)(z^B - z^S) = \overset{0}{\cancel{\Theta_{zz}^{(A)}}} + mR\frac{H}{4} = mR\frac{H}{4}$$

Punkt B liegt nicht auf der Symmetrieachse

$\rightarrow$ Deviationsmoment vorhanden

Starrkörperkinetik in der Ebene



Ebene Bewegung: $\underline{v} = v_x\,\underline{e}_x + v_y\,\underline{e}_y$, $\underline{\omega} = \omega\,\underline{e}_z$

- Impulssatz / Newton: $m\dot{\underline{v}} = \underline{F}$ (Translation)
- Drallsatz / Euler (eben): $\Theta_{zz}^{(S)}\,\dot{\omega} = M^{(S)}$ (Rotation)

kin. Energie des starren Körpers: $E^{kin} = \underbrace{\frac{1}{2}m(v^S)^2}_{\text{Translation}} + \underbrace{\frac{1}{2}\Theta_{zz}^{(S)}\,\omega^2}_{\text{Rotation}}$

- Energiesatz: $E_1^{kin} + E_1^{pot} = E_0^{kin} + E_0^{pot} + \underline{W_{01}^{diss}}$ Arbeit der dissipativen Kräfte und Momente
- Eulersche Kinematikgleichung: $\underline{v}^B = \underline{v}^A + \underline{\omega} \times \underline{x}^{AB}$, mit $\underline{\omega} = \omega\,\underline{e}_z$, $\underline{x}^{AB} = \underline{x}^B - \underline{x}^A$, $\underline{x}^{AB} \cdot \underline{e}_z = 0$

Beispiel Hula Hoop Reifen

nächstes Mal ...