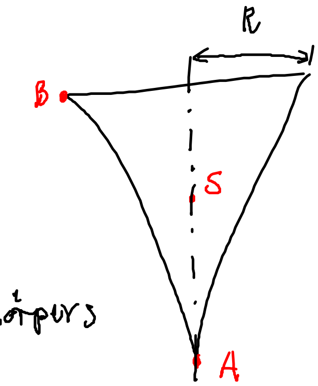
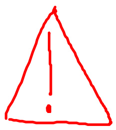
Vorlesung am 29.06.18

- Korrektur zum Satz von Steiner
- Hula hoop Reifen / Gymnastikreifen
- Kreiselgleichungen + Experiment



Satz von Steiner



$$\underline{\underline{\Theta}}^{(A)} = \underline{\underline{\Theta}}^{(S)} + m\left(\underline{x}^{SA}\cdot\underline{x}^{SA}\,\underline{\underline{I}} - \underline{x}^{SA}\otimes\underline{x}^{SA}\right)$$

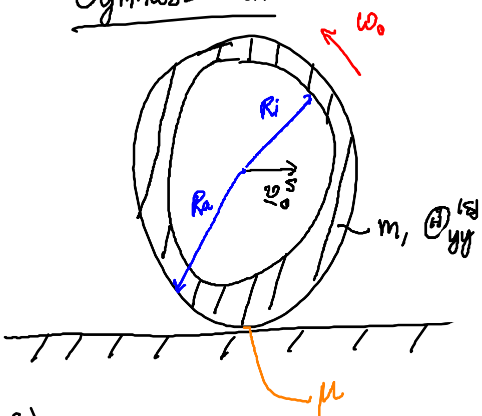
Transformation ist nur mit dem Schwerpunkt S des Körpers möglich!

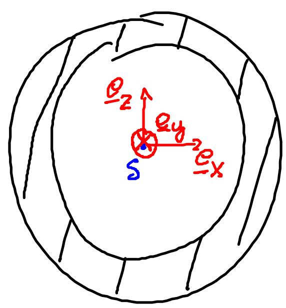
Erlaubt: $\underline{\underline{\Theta}}^{(A)} \longrightarrow \underline{\underline{\Theta}}^{(S)} \longrightarrow \underline{\underline{\Theta}}^{(B)}$

Verboten: $\cancel{\underline{\underline{\Theta}}^{(A)} \longrightarrow \underline{\underline{\Theta}}^{(B)}}$

$$\underline{x}\otimes\underline{x} = \begin{bmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{bmatrix}$$

Gymnastikreifen





Anfangsbedingungen:

$$v^S(t=0) = v_0\,\underline{e}_x$$

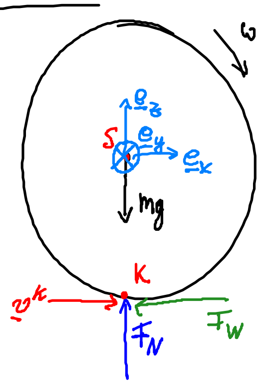
$$\underline{\omega}(t=0) = \underline{\omega}_0 = -\omega_0\,\underline{e}_y$$

Geometrie:

$$\Theta_{yy}^{(S)} = \frac{1}{2}\,m\left(R_a^2 - R_i^2\right)$$

a)

Freischnitt



Impulssatz:

$$m\dot{v}_x = -F_W$$

$$m\dot{v}_z = -mg + F_N$$

Drallsatz:

$$\Theta_{yy}^{(S)}\,\dot{\omega} = F_W\,R_a$$

Unbekannte: $v_x, v_z, F_N, F_W, \omega$ → 5 Unbekannte

Coulomb

$$F_W = \mu F_N$$

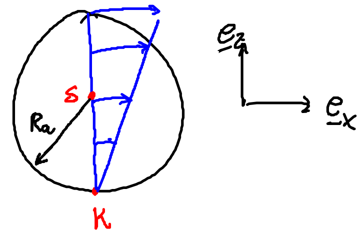
Zwangsbedingung

$$v_z = 0$$

$$\Rightarrow \begin{cases} \cancel{m}\,\dot{v}_x = -\mu \cancel{m}g \\ \frac{1}{2}\cancel{m}\left(R_a^2-R_i^2\right)\dot{\omega} = \mu \cancel{m}g\, R_a \end{cases} \Leftrightarrow \begin{cases} \frac{d v_x}{dt} = -\mu g \\ \frac{d\omega}{dt} = \frac{2R_a}{R_a^2-R_i^2}\,\mu g \end{cases} \Rightarrow \begin{cases} v_x(t) = v_0 - \mu g t \\ \omega(t) = \frac{2R_a}{R_a^2-R_i^2}\,\mu g t - \omega_0 \end{cases}$$

b)

- Bedingung für reines Rollen: $\underline{v}^K = \underline{0}$



$$\underline{v}^K = \underline{v}^S + \underline{\omega} \times \underline{x}^{SK}$$

$$= v_x \underline{e}_x + \omega \underline{e}_y \times (-R_a \underline{e}_z)$$

$$= (v_x - \omega R_a)\,\underline{e}_x$$

→ Zeitpunkt t_R bei dem reines Rollen auftritt: $\underline{v}^K(t_R) = 0 \Leftrightarrow v_x(t_R) = \omega(t_R) R_a$

$$\Leftrightarrow v_0 - \mu g t_R = \frac{2R_a^2}{R_a^2-R_i^2}\,\mu g t_R - \omega_0 R_a$$

$$\Rightarrow t_R = \frac{v_0 + \omega_0 R_a}{\mu g}\;\frac{R_a^2 - R_i^2}{3R_a^2 - R_i^2}$$

- Bedingung für die Bewegungsumkehr: $\underline{v}^S = \underline{0}$

→ Zeitpunkt t_u der Bewegungsumkehr: $\underline{v}^S(t_u) = 0 \Leftrightarrow v_x(t_u) = v_0 - \mu g t_u = 0$

$$\Rightarrow t_u = \frac{v_0}{\mu g}$$

c) Zeitliche Reihenfolge für $R_i = \frac{9}{10} R_a$, $\left(R_i/R_a\right)^2 = \frac{81}{100}$

$$t_R = \frac{v_0 + \omega_0 R_a}{\mu g}\;\frac{1-(R_i/R_a)^2}{3-(R_i/R_a)^2} = \frac{v_0 + \omega_0 R_a}{\mu g}\;\frac{1-\frac{81}{100}}{3-\frac{81}{100}} = \frac{19}{219}\;\frac{v_0 + \omega_0 R_a}{\mu g}$$

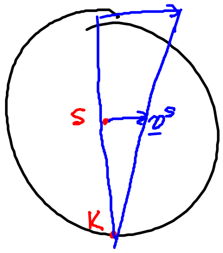
Wann gilt $t_R < t_u$ oder $t_R > t_u$?

$$\frac{19}{219}\;\frac{v_0 + \omega_0 R_a}{\mu g} \begin{cases} < \frac{v_0}{\mu g} \\ > \frac{v_0}{\mu g} \end{cases} \Leftrightarrow \frac{19}{\cancel{219}}\;\frac{\omega_0 R_a}{\cancel{\mu g}} \begin{cases} < \frac{200}{\cancel{219}}\;\frac{v_0}{\cancel{\mu g}} \\ > \frac{200}{\cancel{219}}\;\frac{v_0}{\cancel{\mu g}} \end{cases}$$

$$\omega_0 \begin{cases} < \frac{200}{19}\;\frac{v_0}{R_a} \\ > \frac{200}{19}\;\frac{v_0}{R_a} \end{cases}$$

d) Was passiert falls $t_R < t_u$?

$\underline{t = t_R:}$



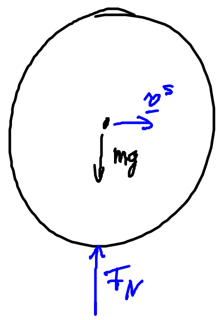
$$v_x(t_R) = v_0 - \mu g t_R = v_0 - \frac{19}{219}\left(v_0 + \omega_0 R_a\right)$$

$$= \frac{200}{219} v_0 - \frac{19}{219} \omega_0 R_a$$

mit $\omega_0 = \alpha \dfrac{200}{19} \dfrac{v_0}{R_a}$, $\alpha \in (0,1)$

$$v_x(t_R) = \frac{200}{219} v_0 - \frac{200}{219} \alpha v_0 = (1-\alpha) \frac{200}{219} v_0 > 0$$

$\underline{\text{Freischnitt}}$



$\underline{\text{Impulssatz:}}$ $m \dot{v}_x = 0 \Rightarrow v_x(t) = v_x(t_R)$

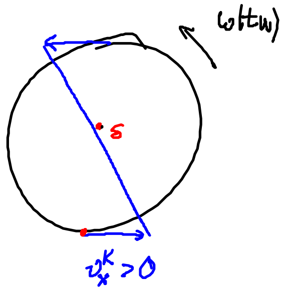
$$= (1-\alpha) \frac{200}{219} v_0$$

Diese Lösung gilt für $t \geq t_R$ und falls $t_R < t_u$.

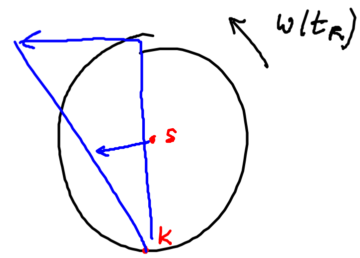
$\rightarrow$ Die Lösung aus a) ist nur für $t < t_R$ gültig!

c) $\underline{t_R > t_u:}$

$\underline{t = t_u:}$

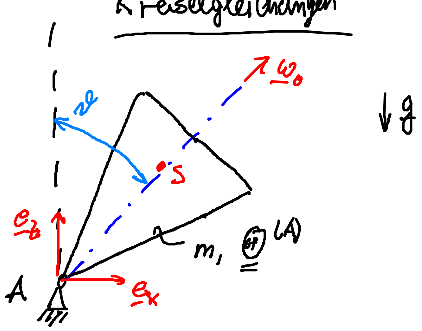


$\underline{t = t_R:}$

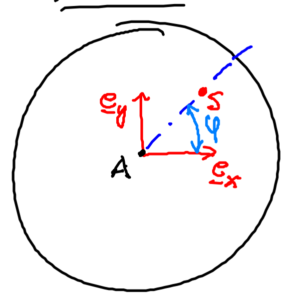


$$\omega(t_R) = 2\mu g \frac{R_a^2}{R_a^2 - R_i^2} t_R - \omega_0 = (v_0 + \omega_0 R_a) \frac{2 R_a}{3 R_a^2 - R_i^2} - \omega_0 = \dots < 0$$

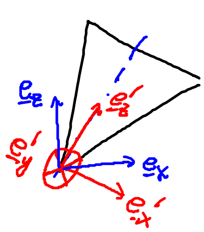
$\underline{\text{Kreiselgleichungen}}$



$\underline{\text{Draufsicht}}$



$\underline{\text{Körperfestes System:}}$



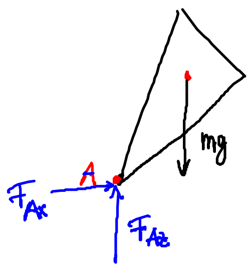
Massenträgheitstensor im Körperfesten ist bekannt!

$$\underline{\underline{\Theta}}^{(A)} = \begin{bmatrix} \Theta_{xx}^{(A)} & 0 & 0 \\ 0 & \Theta_{yy}^{(A)} & 0 \\ 0 & 0 & \Theta_{zz}^{(A)} \end{bmatrix}$$

$$= \Theta_{xx}^{(A)} \underline{e}_x'(t) \otimes \underline{e}_x'(t) + \Theta_{yy}^{(A)} \underline{e}_y'(t) \otimes \underline{e}_y'(t) +$$

$$+ \underline{\underline{\Theta}}_{zz}^{(A)}\, \underline{e}_z'(t) \otimes \underline{e}_z'(t)$$

Freischnitt:



Drallsatz:

$$\frac{d}{dt}\underline{L}^{(A)} = \underline{M}^{(A)}$$

$$\underline{M}^{(A)} = \underline{x}^{AS} \times mg\,\underline{e}_z = \frac{3}{4} mgH\, \underline{e}_y'$$

$$\underline{L}^{(A)} = \underline{\underline{\Theta}}^{(A)} \cdot \underline{\omega}$$

Zeitliche Abhängigkeit der Basisvektoren $\underline{e}_x', \underline{e}_y', \underline{e}_z'$

$$\frac{d\underline{L}^{(A)}}{dt} = \underline{\underline{\Theta}}^{(A)} \cdot \dot{\underline{\omega}} + \underbrace{\underline{\omega} \times (\underline{\underline{\Theta}}^{(A)} \cdot \underline{\omega})}_{\text{Zusatzterm wegen der Relativbewegung}}$$