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An Axiomatic Approach to Classical Dynamics Based on an Objective Power Functional

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Based on an Objective Power Functional

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I. Introduction

The traditional foundations of dynamics are usually laid down by the following assumptions or axioms:

1) There exist preferred frames of observer, called 'inertial frames'.

2) The primitive concepts are forces and torques, each of them an objective vector field. The torques are related to an arbitrary point of reference, and the transformation under change of this point is given in the well-known way, sometimes called Varignon's principle of moments.

3) The balance of momentum and angular momentum with respect to the 'inertial frame' are fulfilled during any possible motion.

In pure theory, primitive concepts do not need a method of measurement ("mental or instrumental") nor a definition. We have either an empirical notion of them from our every-day-life experience, or we obtain them as results from other non-mechanical theories, e.g., astronomy, quantum physics, electrodynamics. But, unfortunately, human creatures have no sense to keep forces and 'inertial forces', i.e., changes of momentum; distinct. And, since we know for sure, that not only the earth, but also the 'fixed' stars, the solar system, and even the galaxies are moving, and thus no astronomer can see an 'inertial frame' in his telescope, we are confronted with the fundamental dilemma, that we do not know what forces are, if we do not know, what an 'inertial frame' is and vice versa. And, moreover, we are far from verifying the two balance laws by experiment, as long as we calculate the forces and torques by them (see TRUESDELL/TOUPIN p. 533). This common practice motivated many authors in this field, to take the two balances as definitions for forces and torques rather than independent laws. By this manipulation the constitutive postulates for forces (see NOLL p. 142), namely, Newton's gravitation law, take over the role of independent laws. Thus, the advance is but none, and the problem with the 'inertial frame' still remains.
Another shortcoming of these theories is the fact, that the fundamental laws of mechanics are not objective. But, this can easily be removed by "sacrificing the objectivity of the external body forces" (NOLL p. 45), and by adding 'inertial forces', if the change of frame is not Galilean. This trick is due to CLAIRAUT, and leads us into the right direction, namely, to sacrifice the entire concept of 'inertial frames'. In the present theory, all frames are equivalent, and the two balances are valid relative to any of them.

It surely was an important discovery by NOLL (p. 140 f.) and GREEN/RIVLIN, to deduce the two balances from the assumption, that the mechanical power is objective.\(^1\)

The next important step was done by GURTIN/WILLIAMS by taking the mechanical power as an objective primitive concept and by deriving the forces, torques, and their balances. Their approach, although mathematically correct and elegant, left some physical questions unsolved. They did not specify the changes of frame, their theory was formulated relative to an absolute space ("classical space-time structure" (NOLL p. 204)), and they assumed, that the power-functional itself would be the same for any frame, and not only it's values for certain arguments. The latter assumption is common to all the literature and caused much confusion and misunderstanding about the principle of material objectivity. And, as will be shown, is too restrictive and cannot be generally expected. Unfortunately, it hides a fundamental dilemma of the power concept, namely, that the forces cannot be uniquely determined, in principle. This is due to the fact, that the velocity at any instant can not be choosen arbitrarily, but instead is determined for continuity reasons, which can not be dropped within Newtonian mechanics resting upon the existance of accelerations.

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\(^1\) In thermodynamics the latter is a consequence of the energy balance and the assumption, that the change of internal energy and the heat supply are objective. See also BEATTY.
Therefore, we have to live with the existence of nondetectable forces and torques, which do not work in any possible motion. But we are already familiar to this fact in the context of internal and external constraints (see TRUESDELL/NOLL p. 70). In fact, this indeterminism does not have any physical significance at all, and all the projectiles, stars, engines, and power stations, that obey the traditional balances, also obey ours, and vice versa. Another fully developed axiomatic theory is that of TROSTEL, who favours inertial frames and starts from other axioms than ours describing the energy of masspoints.

In the next Section we give a short and comprehensive outline of classical, i.e. non-relativistic, kinematics. The leading ideas and results are essentially those of NOLL (p. 204 ff.) and WANG/TRUESDELL. A coordinate-free concept of the space-time continuum, frames, and material bodies is given by the tools of modern differential geometry, such as manifolds, differential forms, and bundles. The reader who is familiar to the subject and accepts the well-known laws of transformation being numbered in the margin, may omit this Section and start with the main concepts of dynamics worked out in Section 3. It starts with a principle of determinism, which states the existence of a balance of an (objective) primitive concept, called power. As an auxiliary concept, the power is extended by means of a virtual power, which enables us to introduce forces and torques by definition. Their transformation properties are studied, and their balance laws are derived from assumptions on the virtual power.

While all these concepts are defined on the moved body as a whole, in Section 4 the field formulations are introduced along the traditional lines of the stress principle of Euler and Cauchy. For shortness, the concepts of measure theory on bodies and sub-bodies (see NOLL p. 32 ff., GURTIN/WILLIAMS, e.g.), that surely play an important role in understanding and describing continuous media, are omitted completely. One of the results in this Section is the objectivity of the stresses. This does not include, that the formula of the stresses is the same for all frames. Within this theory two forms of the so-called principle of material objectivity
(sometimes called the 'active' and the 'passive' version) are kept strictly distinct, being related to different physical phenomena.

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II  Kinematical Preliminaries

An n-dimensional Euclidean space $E^n$ is a metric space with an n-dimensional group of translations $V^n$, which forms a linear space with inner product " $\cdot $ " (see NOLL p. 188 ff. and p. 295). By fixing an arbitrary point $o \in E^n$ as a point of reference, we have an identification between $E^n$ and its translational space $V^n$. For every point $x \in E^n$ there is a unique vector $0x$ in $V^n$ that shifts $o$ into $x$. $0x$ is called the position vector of $x$ relative to $o$.

A Euclidean bundle is a bundle whose base and fibres are Euclidean spaces. More precisely, a Euclidean bundle is a triplet $(\xi, T, \pi)$, such that $\xi$ is a set, called total space, the base space $T$ is Euclidean, and $\pi$ is a surjection

$$\pi : \xi \longrightarrow T,$$

such that for all $t \in T$

$$E_t : = \pi^{-1}(t),$$

are isomorphic Euclidean spaces, called fibres.

A mapping

$$r : T \longrightarrow \xi$$

is called cross-section of the bundle, if

$$\pi \circ r = \text{Id}_T.$$  \(1)\)

holds.

\(1)\) By " $\cdot $ " we denote the composition of functions, and by Id$_T$ the identity mapping of a space $T$. 
The (neo-classical (NOLL p. 204 ff.) ) space-time continuum is a Euclidean bundle \((\mathcal{E},T,t)\), such that the base space is the oriented time continuum \(T\), being isomorphic to the reals \(\mathbb{R}\), and the fibres are 3-dimensional Euclidean spaces, called the instantaneous spaces \(E_t\). The elements of \(\mathcal{E},E_t\) and of \(T\) are called events, instantaneous events, and instants, respectively. The metrics in \(E_t\) assign spatial distances to simultaneous events, the one in \(T\) time differences to (not necessarily simultaneous) events. A spatial distance between non-simultaneous events is not yet defined. A cross-section of the space-time continuum is called path.

A frame (of reference or observer) is a bijection

\[
\phi : \mathcal{E} \longrightarrow \mathbb{R} \times E^3
\]

such that

1.) the restrictions \(1 \circ \phi^{-1} \big|_{\mathbb{R} \times \{x\}} : \mathbb{R} \longrightarrow T\)

are orientation and distance preserving for all \(x \in E^3\);

2.) the restrictions \(\phi^{-1} \big|_{\{t\} \times E^3} : E^3 \longrightarrow E_t\ , t \in \mathbb{R}\)

preserve the spatial distances between simultaneous events.

A frame can always be constituted by a clock

\[
\phi_T : T \longrightarrow \mathbb{R}
\]

and by an indexed set of isometries

\[
\phi_t : E_t \longrightarrow E^3\ , \ t \in T
\]

such that

\[
\phi(\cdot) = (\phi_{\cdot} \cdot \phi(\cdot), \phi_T(\cdot)(\cdot)).
\]

By a frame we can define the spatial distance between events \(e_1\) and \(e_2\) (not necessarily simultaneous) by the distance between \(\phi_{t_1}(e_1)\) and \(\phi_{t_2}(e_2)\) in \(E^3\). Moreover, we can pull back the differentiable structure on \(\mathcal{E}\) by a frame. Let

\[
p_2 : \mathbb{R} \times E^3 \longrightarrow E^3
\]
be the projection in the second argument, and let \( r \) be a path, then the *velocity* of \( r \) is the right-sided derivative of

\[
p_2 \cdot \Phi \circ r \circ \Phi^{-1}_\mathbb{T} : \mathbb{R} \rightarrow \mathbb{E}^3,
\]

if it exists. The second time-derivative is the *acceleration*. A path is called *fixed*, if its velocity is always zero. By the identification between \( \mathbb{E}^3 \) and \( \mathbb{V}^3 \) mentioned above, we can also take the rate of the position vectors with respect to any fixed point of reference. Of course, all these quantities depend on the frame.

If \( \Phi \) and \( \mathbb{V} \) are frames, the mapping

\[
W := \mathbb{V} \circ \Phi^{-1} : \mathbb{R} \times \mathbb{E}^3 \rightarrow \mathbb{R} \times \mathbb{E}^3
\]

is called *change of frame*. For the first argument \( W \) is an orientation and distance preserving mapping \( \mathbb{R} \rightarrow \mathbb{R} \), whereas for the second argument \( W \) can be represented by the autometries

\[
W_t := \mathbb{V}_t \circ \Phi^{-1}_t : \mathbb{E}^3 \rightarrow \mathbb{E}^3,
\]

the index set being the time \( T \). In order to give the transformations of the position vectors, the velocity, and the acceleration, we choose two points of reference \( o_\Phi \) and \( o_\mathbb{V} \) in \( \mathbb{E}^3 \). Let \( r \) be a path, we define

\[
r_\Phi(t) := o_\Phi \circ \Phi^{-1}_t \circ r(t)
\]

and

\[
r_\mathbb{V}(t) := o_\mathbb{V} \circ \mathbb{V}^{-1}_t \circ r(t).
\]

Then there is a unique orthogonal transformation \( Q_t \) on \( \mathbb{V}^3 \), such that

\[
r_\mathbb{V}(t) = Q_t(r_\Phi(t)) + a_t \tag{2.1}
\]
with

\[ a_t := o_y W_t(o_\phi) \quad \text{for all } t \in T. \]

For the velocities and accelerations we obtain the well known formulae

\[ v_y = Q_t(v_\phi) + \dot{Q}_t(r_\phi) + \ddot{a}_t \]
\[ = Q_t(v_\phi) + \omega_t \times (r_y - a_t) + \ddot{a}_t \]

and

\[ b_y = Q_t(b_\phi) + 2\dot{Q}_t(v_\phi) + \ddot{Q}_t(r_\phi) + \dddot{a}_t \]
\[ = Q_t(b_\phi) + \dot{\omega}_t \times (r_y - a_t) - \omega_t \times (\omega_t \times (r_y - a_t)) \]
\[ + 2\omega_t \times (v_y - \dot{a}_t) + \dddot{a}_t, \]

where \( \omega_t \) is the vector belonging to the skew transformation \( \dot{Q}_t \cdot Q_T^t \), such that for each \( v \in \mathbb{R}^3 \)

\[ \omega_t \times v = \dot{Q}_t \cdot Q_T^t (v) \]

holds. Hence, a change of frame induces the vectors \( a_t \) and \( \omega_t \) and the orthogonal transformation \( Q_t \), all of them being time dependent. Note, that only \( a_t \) depends on the choice of the points of reference \( o_\phi \) and \( o_y \), whereas

\[ \omega_t \times (r_y - a_t) + \ddot{a}_t \]

and its time derivative do not.

From now on we restrict our considerations to a (maximal) class of frames \( \Lambda \) such that all changes within this class are two times differentiable, i.e., if the acceleration of a path exists with reference to \( \Phi \in \Lambda \), then it does with reference to each other frame in \( \Lambda \). We call a time-dependent quantity, that is defined for any frame in \( \Lambda \), an observable quantity. For example, the position
vector, the velocity, and the acceleration are observable quantities. We call a scalar-valued function $\alpha$, a vector-valued function $w$, and a tensor-valued function $T$, each of them being an observable quantity, objective under certain/any changes of frame out of $A$, if

$$\alpha_y = \alpha_\phi$$

$$w_y = Q_\tau(w_\phi)$$

$$T_y = Q_\tau \cdot T_\phi \cdot Q_\tau^T$$

hold for the induced orthogonal transformations $Q_\tau$. A change of frame is called Galilean transformation, if all accelerations are transformed like objective observable quantities. This is equivalent to the conditions that $Q_\tau$ and $\dot{a}_\tau$ are time-independent. This notion constitutes an equivalence relation on $A$, and the equivalence classes are called Newtonian spaces.

A body $B$ is a three-dimensional, connected, compact, differentiable manifold. It is endowed with a positive differential-3-form $\omega$, called element of mass. The value $\mathcal{M}(B) = \int \omega$ is called mass of $B$. A body $B_1$ is called a subbody of another body $B_2$, if $B_1$ is an (open) submanifold of $B_2$ and if $\omega_1$ is the restriction of $\omega_2$ to $B_1$.

A motion $\kappa$ of a body $B$ is a mapping

$$\kappa : T \times B \rightarrow \mathcal{C}$$

such that

1.) for all $P \in B$, $\kappa(\cdot, P) \rightarrow \mathcal{C}$ is a twice piecewise countinously differentiable path;

2.) for all $t \in T$, $\kappa(t, \cdot)$ is an imbedding of $B$ in $E_t$, and the frontier $V := \partial \kappa(t,B)$ is regular (in order to apply the divergence theorem).

We write $\overline{\kappa(t,B)} := \kappa(t,B) \cup \partial \kappa(t,B)$ for the closure, where $E_t$ is endowed with the initial topology of $E^3$ under any $\phi \in A$. 
By these imbeddings we can transform any function defined on $\{t\} \times \mathcal{B}$ (material description) likewise on $\kappa(t, B)$ (spatial description), and vice versa, which will be done without being mentioned.

The class of motions that a body can perform is an individual property of that body. If we talk about a motion later on, we will tacitly assume, that it is in this class, which may be restricted by internal or external constraints as well as dynamical restrictions. A motion of a body is called rigid, if for all $t \in T$ the spatial distance between $\kappa(t, P_1)$ and $\kappa(t, P_2)$ is constant for all $P_1, P_2 \in \mathcal{B}$. For rigid motions the formula of Euler holds:

$$v_\phi(t, P_2) = v_\phi(t, P_1) + \Omega_\phi(t) \times r_{12}$$

with

$$r_{12} := \frac{\Phi_t \cdot \kappa(t, P_1) - \Phi_t \cdot \kappa(t, P_2)}{r_{12}}.$$

The vector $\Omega_\phi(t)$ is called angular velocity. It is an observable quantity and transforms like

$$\Omega_\phi(t) = Q_t(\Omega_\phi(t)) + \omega_t.$$

Let

$$V_\phi := \Phi_t \cdot \kappa(t, B)$$

and

$$D_\phi := D(\Phi_t \cdot \kappa(t, B)).$$

Two more vector valued observable quantities will be needed later: the momentum

$$I_\phi(t, B) := \int_{\mathcal{V}_\phi} v_\phi(t, x) \, dm$$

the angular momentum: with respect to a fixed point of reference o

$$D_\phi(t, B) := \int_{\mathcal{V}_\phi} \bar{\omega} \times v_\phi(t, x) \, dm.$$
Of course, their time derivatives are

\[ \dot{I}_\phi(t, B) := \int_{V_\phi} b_\phi(t, x) \, dm \]  

(2.8)

and

\[ \dot{D}_\phi(t, B) := \int_{V_\phi} \vec{\omega} \times b_\phi(t, x) \, dm \]

(2.8)

III Dynamics of Continuous Bodies

In this Section we consider an arbitrary body \( B \), an arbitrary, but fixed motion \( \kappa \), and a fixed instant \( t \), if not explicitly stated otherwise. The frames are indicated by suffixes. The velocity field on \( \Phi \cdot \kappa(t, B) \) is denoted by \( v_\phi \), which is in general time dependent.

Axiom. Principle of determinism.

For every body under any motion there exists a real valued observable quantity \( L_\phi \), called power, which obeys at all times the balance of power

\[ L_\phi = 0 \]  

(3.1)

with respect to any frame \( \phi \).

In general, the power is a functional of the entire motion. For philosophical or technical reasons its domain can be restricted on the past or on a finite time interval, but this is not necessary for our analysis.

Note, that by the balance of power the latter is objective.
Let $\delta V_\phi$ denote the set of vector fields defined on $\phi(t, y)$. By pointwise linear operations $\delta V_\phi$ is a infinite-dimensional vector space. If we extend the velocity field continuously on $0_\phi$, then it is element of $\delta V_\phi$.

**Definition.** A **virtual power** is a linear objective extension of the power on $\delta V_\phi$, i.e. a time-dependent function

$$\delta L_\phi : \delta V_\phi \rightarrow \mathbb{R}, \quad (3.2)$$

such that

(V1) $\delta L_\phi$ is linear.

(V2) $\delta L_\phi(v_\phi) = L_\phi$, if $v_\phi$ is the extended velocity field.

(V3) $\delta L_\phi(w_\phi) = \delta L_\psi(w_\psi)$ \hspace{1cm} (3.3)

if $w$ transforms like velocities:

$$w_\psi = Q_\psi(w_\phi) + \dot{\omega}_t \times (r_\psi - a_t) + \ddot{a}_t. \quad (3.4)$$

A virtual power always exists, at least as the degenerate zero function. The problem which will be considered later is that there are infinitely many virtual powers.

The following Proposition follows from the fact, that every linear functional can be represented by a dual vector.

**Proposition and Definition.** 3.1. Let $\delta L$ be a virtual power. Then there exist two vector valued observable quantities $f_0$ and $m_0$ such that for every rigid modification

$$w_\phi = v_\phi + u_0_\phi + \Omega_\phi \times r_0_\phi \quad (3.5)$$

the virtual power can be represented by
\[ \delta L_\phi(w_\phi) = L_\phi + f_\phi \cdot u_\phi + m_\phi \cdot \Omega_\phi. \]  

(3.6)

We call \( f_\phi \) and \( m_\phi \) the lost force and the lost torque, respectively, induced by the virtual power \( \delta L \).

Note, that \( f_\phi \) and \( m_\phi \), as well as \( L_\phi \), can be functions of the entire motion \( \hat{\kappa} \).

Definition. Let \( I_\phi \) be the momentum and \( D_\phi \) the angular momentum. We call the time-dependent vector-valued quantities

\[ K_{O_\phi} := f_{O_\phi} + I_\phi \]  

(3.7)

the force, and

\[ M_{O_\phi} := m_{O_\phi} + D_\phi \]  

(3.8)

the torque induced by the virtual power \( \delta L \).

In the foregoing Definition the suffices stand for the point of reference and for the frame, both being completely arbitrary. We will now clarify the dependence of the first.

Proposition. 3.2. Let \( O_\phi \) and \( O_\phi' \) be two points of reference. Then the following transformations hold for the lost forces, lost torques, forces, and torques induced by a virtual power:

\[ f_{O_\phi'} = f_{O_\phi} \]  

(3.9a)

\[ m_{O_\phi'} = m_{O_\phi} + O_\phi' \times f_{O_\phi} \]  

(3.9b)

\[ K_{O_\phi'} = K_{O_\phi} \]  

(3.9c)

\[ M_{O_\phi'} = M_{O_\phi} + O_\phi' \times K_{O_\phi} \]  

(3.9d)
Proof: All the following quantities are taken with respect to any frame. By Euler's formula (2.5) we have

\[ v_0 + \Omega_o \times o\mathbf{x} = v_{0'} + \Omega_{0'} \times o'\mathbf{x}. \]

This holds for all \( x \in \phi_{t-k}(t, B) \), if, and only if

\[ v_{0'} = v_0 + \Omega_o \times oo' \]

and

\[ \Omega_{0'} = \Omega_o \]

hold. According to Proposition (3.1) we have

\[ \delta L(v_0 + \Omega_o \times o\mathbf{x}) \]

\[ = f_o \cdot v_o + m_o \cdot \Omega_o \]

\[ = \delta L(v_{0'} + \Omega_{0'} \times o'\mathbf{x}) \]

\[ = f_{0'} \cdot v_{0'} + m_{0'} \cdot \Omega_{0'} \]

\[ = f_{0'} \cdot v_o + f_{0'} \cdot (\Omega_o \times oo') + m_{0'} \cdot \Omega_o \]

\[ = f_{0'} \cdot v_o + (m_{0'} + oo' \times f_{0'}) \cdot \Omega_o. \]

By the arbitrariness of \( v_o \) and \( \Omega_o \), (a) and (b) follow. (c) is a direct consequence of (a) and the fact, that \( \mathbf{I} \) does not depend on a point of reference. By (3.8) we have

\[ M_{0'} = m_{0'} + D_{0'} \]

\[ = m_o + oo' \times f + \int_{oo'} x \times \dot{v} \, dm \]

\[ = m_o + oo' \times (f + \int \ddot{v} \, dm) + \int_{oo'} \dot{v} \, dm \]

\[ = m_o + oo' \times K + D_o \]

\[ = M_o + oo' \times K; \quad q.e.d. \]
Accordingly, we can suppress the index of the point of reference for the lost forces and the forces, and, henceforth, we write $f_\phi$ and $K_\phi$.

**Theorem. 3.3. D'Alembert's principle.** The lost forces and the lost torques induced by any virtual power are balanced with respect to every frame $\phi$

$$f_\phi = 0 \quad (3.10)$$

$$m_{0\phi} = 0 \quad (3.11)$$

This is equivalent to the balance of momentum

$$K_\phi = \mathbf{l}_\phi \quad (3.12)$$

and the balance of angular momentum

$$M_{0\phi} = D_{0\phi} \quad (3.13)$$

**Proof.** By (V1), (V3), and Proposition 3.1 we obtain for any frames $\phi$ and $\psi$ and for every vector field $\mathbf{w}_\phi \in \delta V_\phi$

$$\delta L_\psi(Q_t(\mathbf{w}_\phi)) + \omega_t \times \mathbf{r}_{0\psi} + \dot{\mathbf{a}}_t = \delta L_\phi(\mathbf{w}_\phi)$$

$$= \delta L_\psi(Q_t(w_\phi)) + \delta L_\psi(\omega_t \times \mathbf{r}_{0\psi}) + \delta L_\psi(\dot{\mathbf{a}}_t)$$

$$= \delta L_\psi(Q_t(\mathbf{w}_\phi)) + f_\psi \cdot \dot{\mathbf{a}}_t + m_{0\psi} \omega_t .$$

If we choose $\mathbf{w}_\phi = 0$, the last two terms must vanish. As $\phi$ was an arbitrary frame, $\dot{\mathbf{a}}_t$ and $\omega_t$ can have any value independent of $f_\psi$ and $m_{0\psi}$, and, hence, we conclude (3.10) and (3.11). The equivalence to the two balances follows immediately by the definitions of force and torque, q.e.d.
Clearly, the lost forces and lost torques are objective. In the next Proposition we choose the points of reference for simplicity such that \( \Phi_t^{-1}(\phi_0) = \Psi_t^{-1}(\psi_0) \). This seems rather special, but in connections with Proposition (3.2) we obtain full generality.

**Proposition. 3.4.** Transformation laws of forces and torques under change of frame.

Let

\[
\Phi_t^{-1}(\phi_0) = \Psi_t^{-1}(\psi_0),
\]

then

\[
K_\psi = Q_t(K_\phi) + \int_{V_\psi} e_t \, dm
\]  \hspace{1cm} (3.14)

and

\[
M_{\phi \psi} = Q_t(M_{\phi \phi}) + \int_{V_\psi} r_{\phi \psi} \times e_t \, dm
\]  \hspace{1cm} (3.15)

hold with

\[ e_t := \dot{\omega}_t \times r_{\phi \psi} - \omega_t \times (\omega_t \times r_{\phi \psi}) + 2\omega_t \times (v_\psi - \dot{a}_t) + \ddot{a}_t. \]  \hspace{1cm} (3.16)

**Proof.** By (3.12) and (2.3) we obtain

\[
K_\phi = \int_{V_\phi} b_\phi \, dm
\]

and

\[
K_\psi = \int_{V_\psi} b_\psi \, dm = \int_{V_\psi} \{Q_t(b_\phi) + e_t\} \, dm
\]

\[
= Q_t(I_\phi) + \int_{V_\psi} e_t \, dm = Q_t(K_\phi) + \int_{V_\psi} e_t \, dm.
\]

By (3.13) and (2.1) we obtain

\[
M_{\phi \phi} = \int_{V_\phi} r_{\phi \phi} \times b_\phi \, dm
\]
and

\[ M_{o_y} = \int r_{o_y} \times \{ Q_t(b_y) + e_t \} \, dm \]

\[ = Q_t \left( \int r_{o_y} \times b_y \, dm \right) + \int r_{o_y} \times e_t \, dm \]

\[ = Q_t(M_{o_y}) + \int r_{o_y} \times e_t \, dm ; \text{q.e.d.} \]

We avoid the name 'apparent' or 'inertial' force and torque for the last terms in (3.14) and (3.15), respectively, because no frame is preferred in this text. If we restrict the foregoing results to Galilean transformations, the term \( e_t \) vanishes. Hence, we obtain

**Proposition 3.5.** Forces and torques are objective under Galilean transformations.
IV. Field Theory

Until now, forces and torques are global concepts, i.e. defined on the body itself. In order to achieve a field formulation we follow the traditional lines, which are to be found in most books on continuum mechanics under the label 'stress principle of Euler and Cauchy'. It postulates that the forces and torques arise from contact parts \( K_C \), \( M_C \) and body parts \( K_m \), \( M_m \), each of which is absolutely continuous and thus can be integrated by means of densities over the surface

\[
0_\phi := \delta(\phi_t \cdot \kappa(t,B)) \tag{4.1}
\]

and the interior

\[
V_\phi := \phi_t \cdot \kappa(t,B) \tag{4.2}
\]

of the body in the actual placement. We assume the densities to be continuous. All of the following quantities are observable and, therefore, depend on the frame, to which they are related:

\[
K = K_C + K_m = \int_{0}^{V} t(x) \, d0 + \int_{V} p(x) \, dm \tag{4.3}
\]

\[
M_C = M_C + M_m = \int_{0}^{V} c_0(x) \, d0 + \int_{V} h_0(x) \, dm \tag{4.4}
\]

Under additional assumptions we can follow the concept of the stress tensor \( T(t,x) \) and the couple stress tensor \( C(t,x) \), which serve for

\[
t = T(n) \tag{4.5}
\]

and

\[
c = C(n) \tag{4.6}
\]

where \( n \) is the outward normal of the body surface. We took \( x \) as a point of reference for \( c \).

It would be interesting to find out, if the stress principle of Euler and Cauchy can be replaced by an analogous assumption in terms of (virtual) power instead of forces and torques.
Proposition 4.1. Let \( W_t = \psi_t \cdot \phi_t^{-1} \) be a change of frame, determining the quantities \( q_t \), and \( e_t \) from (3.16). Then the following transformations hold:

a) contact forces and couples are objective:

\[
K_C \psi = Q_t(K_C \phi) \tag{4.7a}
\]
\[
M_{C0\psi} = Q_t(M_{C0\phi})
\]

if
\[
\psi_t^{-1}(o_\psi) = \phi_t^{-1}(o_\phi)
\]

b) their densities are objective:

\[
t_\psi = Q_t(t_\phi) \tag{4.7b}
\]
\[
c_\psi = Q_t(c_\phi)
\]

c) the stress tensor and the couple stress tensor are objective:

\[
T_\psi = Q_t \cdot T_\phi \cdot Q_t^T \tag{4.7c}
\]
\[
C_\psi = Q_t \cdot C_\phi \cdot Q_t^T
\]

d)  
\[
K_m \psi = Q_t(K_m \phi) + \int_{V_\psi} e_t(x) \, dm \tag{4.7d}
\]
\[
M_{m0\psi} = Q_t(M_{m0\phi}) + \int_{\Gamma_\psi} \overrightarrow{\gamma} x \times e_t(x) \, dm
\]

e)  
\[
P_\psi = Q_t(p_\phi) + e_t(x) \tag{4.7e}
\]
\[
h_\psi = Q_t(h_\phi) + \overrightarrow{o_\psi x} \times e_t(x)
\]
Proof. By Proposition 3.4, we have

\[ K_\psi = Q_t(K_\phi) + \int_{V_\psi} e_t(x) \, dm \]

\[ = K_{c\psi} + K_{m\psi} = Q_t(K_{c\phi}) + Q_t(K_{m\phi}) + \int_{V_\psi} e_t(x) \, dm \]

\[ = \int_{0_\psi} t_\psi dO + \int_{V_\psi} P_\psi \, dm \]

\[ = \int_{0_\psi} Q_t(t_\phi) \, dO + \int_{V_\psi} \{Q_t(p_\phi) + e_t\} \, dm \, . \]

This equation must hold for the body \( B \) as well as for any subbody. By the continuity of the densities we conclude the first equations of b) and e). By integration we obtain the first equations of a) and d). The second equations of a), b), d) and e) can be proved analogously. c) is a consequence of b), if we use the fact, that the outward normal vector of a surface, on which \( T \) and \( C \) act, is objective; q.e.d.

Part c) of the foregoing Proposition is usually assumend as an axiom, called the principle of material objectivity or of material indifference. The version given here is weaker than the usual one, as we do not assume that the functions for \( T \) and \( C \) are the same for every frame. If we write down c) with all of its arguments, we obtain

\[ T_\psi(\psi \cdot \kappa) = Q_t \cdot T_\phi(\phi \cdot \kappa) \cdot Q_t^T \tag{4.8} \]

and

\[ C_\psi(\psi \cdot \kappa) = Q_t \cdot C_\phi(\phi \cdot \kappa) \cdot Q_t^T \, . \tag{4.9} \]

Hence, we consider one body under one motion by two different frames. This principle does not restrict the stress function with respect to one frame at all, but if it is known for one frame, it is determined for every other frame by (4.8) and (4.9). This con-
cept shall be kept distinct from the following condition which, as a material property, is obeyed by certain materials, and by other materials fails.

Here, we consider one body under two motions by one frame.

**Condition of indifference under rigid modifications (IRM).**

Let \( \kappa \) be a motion of a body and \( Q_t \) an orthogonal tensor. Then there exists a rigid modification \( \alpha \) of \( \kappa \), such that

\[
Q_t = \text{grad}(\Phi \cdot \alpha \cdot \Phi^{-1}),
\]

(4.10)

and \( \alpha \cdot \kappa \) is another possible motion, at least as a restriction to a subbody containing the point under consideration. Moreover,

\[
T_\Phi(\Phi \cdot \alpha \cdot \kappa) = Q_t \cdot T_\Phi(\Phi \cdot \kappa) \cdot Q_T \]

(4.11)

and

\[
C_\Phi(\Phi \cdot \alpha \cdot \kappa) = Q_t \cdot C_\Phi(\Phi \cdot \kappa) \cdot Q_T
\]

(4.12)

hold for every frame \( \Phi \).

**Proposition.** 4.2. The IRM-condition holds for a material, if, and only if

\[
T_\Phi(\cdot) = T_\Phi(\cdot)
\]

(4.13)

and

\[
C_\Phi(\cdot) = C_\Phi(\cdot)
\]

(4.14)

hold for all frames \( \Phi \) and \( \Psi \).

**Proof.** For every rigid modification \( \alpha \) we can find a frame \( \Psi \) such that \( \Psi = \Phi \cdot \alpha \). By (4.8) and (4.11) we obtain

\[
T_\Phi(\Psi \cdot \kappa) = T_\Phi(\Phi \cdot \alpha \cdot \kappa) = Q_t \cdot T_\Phi(\Phi \cdot \kappa) \cdot Q_T = T_\Psi(\Psi \cdot \kappa).
\]

The proof for the couple stress tensor is entirely analogous; q.e.d.
In other texts it is always assumed that the constitutive functionals are the same to whatever frame they are related, and hence, the principle of material objectivity and the IRM-condition cannot be distinguished. But within the present theory, we are able to construct stress functionals that depend, e.g., on the angular velocity of the body relative to the frame. This does not contradict the principle of material objectivity, but it does contradict the IRM-condition. (Such materials were investigated by BURNETT, MÜLLER, EDELEN /McLENNAN and others).

The so-called 'reduced constitutive equations' (TRUEDELL/NOLL p. 66 ff.) can not be obtained by the principle of objectivity alone, but in connections with the validity of the IRM-condition.

We now restrict our considerations to the non-polar case

$$h = 0 = c \rightarrow T = T^T.$$  \hspace{1cm} (4.15)

If we put $t = T(n)$ into (3.12) and use the divergence theorem, we obtain the well known equation of motion

$$\text{div } T + \rho p = \rho \dot{b}.$$  \hspace{1cm} (4.16)

By multiplication with any field $w \in \delta V$ and by integration we arrive at

$$\int_V \{ \frac{1}{\rho} (\text{div } T) \cdot w + p \cdot w - b \cdot w \} \, dV = 0 \hspace{1cm} (4.17)$$

and using the divergence theorem again, we finally obtain

$$\delta L_0(w) := \int_{\Omega} t \cdot w \, d\Omega + \int_{\partial V} p \cdot w \, dm - \int_{\partial V} b \cdot w \, dm - \int_{\Omega} \text{tr}(T \cdot \nabla w) \, dV = 0.$$  \hspace{1cm} (4.18)
We define this function for other frames by using (V3) and by transforming \( w \) like a velocity (3.4). This function has the following properties:

- \( \Delta L_0(v) = 0 = L \), if \( v \) is the velocity field;
- \( \Delta L_0(w_o) = \Delta L(w_o) \), if \( w_o \) is any constant vector field;
- \( \Delta L_0(\Omega \times \vec{\Omega}) = \Delta L(\Omega \times \vec{\Omega}) \) for any vector \( \Omega \);
- \( \Delta L_0 : \Delta V \rightarrow \mathbb{R} \) is linear.

i.e. \( \Delta L_0 \) is a virtual power, which induces the same force and torque as the original one \( \Delta L \) does, as \( \Delta L_0 \) coincides with \( \Delta L \) on the set of all rigid modifications of the velocity. For other fields in \( \Delta V \), \( \Delta L \) and \( \Delta L_0 \) can be different. But for these arguments the virtual power has no consequences or physical relevance at all.

Let us denote the set of all rigid modifications of the velocity field in \( \Delta V \) by \( \Delta V_o \), i.e.

\[
\Delta V_o := \{ w \in \Delta V | w = v + v_o + \Omega \times \vec{\Omega}, \Omega \in \mathbb{E}^3, v_o \in \mathbb{E}^3 \} 
\tag{4.19}
\]

Then

\[
\Delta L_0 | \Delta V_o = \Delta L | \Delta V_o \tag{4.20}
\]

holds. This gives rise to an equivalence relation on the set of all virtual powers, which can be made for polar or non-polar materials:

\[
\Delta L_1 \sim \Delta L_2 : \Delta L_1 | \Delta V_o = \Delta L_2 | \Delta V_o \tag{4.21}
\]

We state the foregoing results in
Proposition. 4.3. Two virtual powers are equivalent, if, and only if, they induce the same dynamical quantities (forces, torques, lost forces, lost torques).

Proposition. 4.4. For non-polar materials let $\delta L$ be a virtual power and let $\delta L_o$ be defined by (4.18). Then $\delta L$ and $\delta L_o$ are equivalent.

$\delta V_o$ is only a linear subspace of $\delta V$, if the velocity is zero, but, unfortunately, this property is not frame indifferent. We can always shift the origin of $\delta V$ into the velocity field. Then $\delta V_o$ becomes a six-dimensional linear subspace of $\delta V$ which is always contained in the kernel of every virtual power. In the light of Proposition (4.3) we could have restricted our considerations from the outset to virtual powers defined on $\delta V_o$ instead of $\delta V$. But its linearity can only be defined for the rather special case when $v = 0$ and $\delta V_o$ is a linear subspace. This is why we did not do it.

V. Concluding Remarks

One of the results of the foregoing analysis is that the forces and torques cannot be determined uniquely by only measuring the mechanical power and the geometrical/chronometrical quantities. But, moreover, there seems to be no need for such a determinism, as all the different virtual powers induce dynamical quantities which fulfill the dynamical balances altogether. Thus, what we are doing using our traditional dynamical quantities, is like working on representitives, a procedure which shall not be criticized here. This situation reminds us to the fact, that the entropy within the context of thermodynamics is also only a representative (see MEIXNER, KERN, DAY).
References:


